

The irreducibility of the spaces of rational curves on del Pezzo surfaces

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Abstract

Let X be a del Pezzo surface of degree d , and assume that X is general if $d = 1$. We prove that the spaces $\mathcal{M}_{0,0}(X, \beta)$ are either empty or irreducible, if $(d, \beta) \neq (1, -K_X)$. When $(d, \beta) = (1, -K_X)$ it is well known that $\mathcal{M}_{0,0}(X, \beta)$ consists of twelve reduced points.

Introduction

Let X be a del Pezzo surface. If the degree of X is one, assume that X is general. Let $\beta \in H_2(X, \mathbb{Z})$ be the class of a curve on X . Denote by $R(\beta)$ the subscheme of the linear system $|\beta|$ consisting of the integral nodal curves of geometric genus zero. The Kontsevich mapping space $\overline{\mathcal{M}}_{0,0}(X, \beta)$ is a natural compactification of the space $R(\beta)$. Some care is required, since the mapping spaces in general have more irreducible components than the corresponding spaces $R(\beta)$, arising from degenerate configurations of curves on the surface. In fact it may happen that $R(\beta) = \emptyset$, while $\overline{\mathcal{M}}_{0,0}(X, \beta) \neq \emptyset$.

The mapping spaces parametrize the set of all (stable) maps to the surface X from possibly reducible curves. The domain curves of the maps in $\overline{\mathcal{M}}_{0,0}(X, \beta)$ are connected and nodal, have all components isomorphic to \mathbb{P}^1 and the components are attached in such a way that the resulting topological space is simply connected. We refer to such a domain curve as a “rational tree.” Taking the image of a map yields a morphism FC from (the semi-normalization of) $\overline{\mathcal{M}}_{0,0}(X, \beta)$ to the closure in $|\beta|$ of $R(\beta)$ (see [Ko] Section I.6).

Let $\overline{\mathcal{M}}_{bir}(X, \beta)$ be the closure of the subspace of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ consisting of morphisms $f : C \rightarrow X$, with $C \simeq \mathbb{P}^1$ and f birational onto its image. In many interesting cases it is true and easy to check that the map FC defined above is in fact birational, when restricted to $\overline{\mathcal{M}}_{bir}(X, \beta)$.

The main result of this paper is that the space $\overline{\mathcal{M}}_{bir}(X, \beta)$ is irreducible or empty, except in the case where X has degree one and $\beta = -K_X$.

The idea of the proof is straightforward. First, prove that in the boundary of all the irreducible components of $\overline{\mathcal{M}}_{bir}(X, \beta)$ there are special morphisms of a given type (called in what follows “morphisms in standard form”). Second, show that the locus of morphisms in standard form is connected and contained in the smooth locus of $\overline{\mathcal{M}}_{bir}(X, \beta)$.

From these two facts we conclude immediately that the smooth locus of $\overline{\mathcal{M}}_{bir}(X, \beta)$ is connected. Since the smooth locus is dense, we deduce that $\overline{\mathcal{M}}_{bir}(X, \beta)$ is irreducible.

The methods used in the proof are of two different kinds. First, there are general techniques, mainly Mori’s Bend and Break Theorem, to break curves into components with low anticanonical degree. In the case where X is the projective plane, this shows that we may specialize a morphism in $\overline{\mathcal{M}}_{bir}(X, \beta)$ so that its image is a union of lines. Second, we need explicit geometric arguments to deal with the low degree cases. Again, in the case of the projective plane, this step is used to bring the domain to a standard form (a chain of rational curves, rather than a general rational tree), while preserving the property that the image of the morphism consists of a union of lines.

To analyze the curves of low anticanonical degree on a del Pezzo surface, we need a detailed description of their divisor classes in $\text{Pic}(X)$. In particular, we use the group of symmetries of the Picard lattice to reduce the number of cases to treat. Section 2.3 is devoted to this analysis.

Two technical deformation-theoretic tools prove useful. The first is a description of the obstruction space of a stable map to a smooth surface in terms of combinatorial invariants of the map. This is proved in Section 1.2. The second is a lifting result that allows us, given a deformation of a component of a curve, to get a deformation of the whole curve. The statement is proved in Lemma 1.28 and the construction following it is the way in which we are typically going to use it. This is specific to the surface case. The lifting result allows us to deform a map with reducible domain by deforming only a few components at a time. This is done systematically in Section 3.1. We are therefore able to reduce the general problem to relatively few special cases, cf. Theorem 4.5. The explicit computation of the obstruction spaces allows us to prove that in the deformations performed we never move to a different irreducible component of the moduli space.

The connectedness of the locus of morphisms in standard form is a consequence of some explicit computations, some of which are reformulations of classical geometric statements, such as the fact that the ramification locus

of the projection from a general point on a smooth cubic surface in \mathbb{P}^3 is a smooth plane quartic curve. This is the content of Section 4.2, but see also Section 4.1.

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1 Cohomology Groups and Obstruction Spaces

1.1 Rational Trees

The purpose of this section is to prove some general results which are useful to compute the cohomology groups of coherent sheaves on rational trees.

Definition 1.1 *A rational tree C is a connected, projective, nodal curve of arithmetic genus zero. If C is a rational tree, we call a component E of C an end if E contains at most one node of C .*

Definition 1.2 *Given a connected projective nodal curve C , define the dual graph of C to be the graph Γ_C whose vertices are indexed by the components C_i of C and whose edges between the distinct vertices $[C_i]$ and $[C_j]$ are indexed by $\{p \in C_i \cap C_j\}$.*

Remark. A connected projective nodal curve C is a rational tree if and only if all its components are smooth rational curves and its dual graph Γ_C is a tree (for a proof see [De]).

Lemma 1.3 *Let C be a rational tree, and let $\nu : \tilde{C} \rightarrow C$ be the normalization of C at the points $\{p_1, \dots, p_r\} \subset \text{Sing}(C)$; denote by $\iota : \{p_1, \dots, p_r\} \hookrightarrow C$ the inclusion morphism. For any locally free sheaf \mathcal{F} of finite rank on C we have the following short exact sequence of sheaves on C :*

$$0 \longrightarrow \mathcal{F} \longrightarrow \nu_* \nu^* \mathcal{F} \longrightarrow \iota_* \mathcal{F}|_{\{p_1, \dots, p_r\}} \longrightarrow 0$$

Remark. From now on, we may sometimes denote the sheaf $\iota_* \mathcal{F}|_{\{p_1, \dots, p_r\}}$ simply by $\oplus \mathcal{F}_{p_i}$, and similarly for the pushforwards of sheaves on irreducible components of a curve.

Proof. Consider the sequence defining the sheaf \mathcal{Q} :

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_* \mathcal{O}_{\tilde{C}} \longrightarrow \mathcal{Q} \longrightarrow 0 \quad (1.1)$$

Since ν is an isomorphism away from the inverse image of the p_i 's, it follows that \mathcal{Q} is supported at the union of the p_i 's. We now want to prove that \mathcal{Q}_{p_i} is a skyscraper sheaf, i.e. that it has length one. Since this is a local property, it is enough to check it when C has a unique node. In this case, C is the nodal union of two smooth \mathbb{P}^1 's, and \tilde{C} is their disjoint union. Since the normalization map is finite, it is affine, and therefore $H^j(\nu_* \mathcal{O}_{\tilde{C}}, C) \simeq H^j(\mathcal{O}_{\tilde{C}}, \tilde{C})$. Therefore the long exact sequence defining \mathcal{Q} is given by

$$0 \longrightarrow k \longrightarrow k + k \longrightarrow \mathcal{Q} \longrightarrow 0$$

and we deduce that the length of \mathcal{Q} is 1. Thus it follows in general that $\mathcal{Q} = \oplus \mathcal{O}_{p_i}$, the direct sum of the skyscraper sheaves of the nodes p_1, \dots, p_r .

Let us now go back to the sequence (1.1). Since the sheaf \mathcal{F} is locally free, we may tensor the sequence by \mathcal{F} , preserving exactness. To identify the tensor product in the middle we use the projection formula:

$$\nu_* \mathcal{O}_{\tilde{C}} \otimes \mathcal{F} \cong \nu_*(\mathcal{O}_{\tilde{C}} \otimes \nu^* \mathcal{F}) \cong \nu_*(\nu^* \mathcal{F})$$

and we may therefore write the tensored sequence as

$$0 \longrightarrow \mathcal{F} \longrightarrow \nu_* \nu^* \mathcal{F} \longrightarrow \bigoplus_i \mathcal{F}_{p_i} \longrightarrow 0$$

thus proving the lemma. \square

Given a rational tree C and a node $p \in C$, construct a new curve C' as follows: consider the normalization of $\nu : \tilde{C} \rightarrow C$ of C at the point p , and let $\{p_1, p_2\} = \nu^{-1}(p)$. Attach to \tilde{C} a smooth rational curve E so that $\tilde{C} \cap E = \{p_1, p_2\}$ and $C' := \tilde{C} \cup E$ is a nodal curve. Clearly we have a morphism $\pi : C' \rightarrow C$, which is an isomorphism away from E and contracts E to the node p . We call the morphism π the contraction of E . The curve C' so obtained is called the “total transform” of C at the node p and E the “exceptional component.”

Lemma 1.4 *Let C be a rational tree, and let \mathcal{F} be a locally free coherent sheaf on C . Let $\pi : C' \rightarrow C$ denote the total transform of C at a node $p \in C$; then $H^1(C, \mathcal{F}) \cong H^1(C', \pi^* \mathcal{F})$.*

Proof. The result follows immediately from the Leray’s spectral sequence associated to the map π and the fact that $R^i \pi_*(\mathcal{O}_{C'}) = 0$, for $i > 0$. \square

Lemma 1.5 *Let C be a rational tree and let \mathcal{F} be a locally free sheaf on C . Suppose that $C = C_1 \cup C_2$, where C_1 and C_2 are unions of components having no components in common. Let $\{p_1, \dots, p_r\} = C_1 \cap C_2$ be the nodes of C lying on C_1 and C_2 . If $h^1(C_1, \mathcal{F}|_{C_1}(-p_1 - \dots - p_r)) = 0$, then $H^1(C, \mathcal{F}) \cong H^1(C_2, \mathcal{F}|_{C_2})$.*

Proof. Simply consider the long exact sequence associated to the “component sequence”

$$0 \longrightarrow \mathcal{F}|_{C_1}(-p_1 - \dots - p_r) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}|_{C_2} \longrightarrow 0$$

where the first map is extension by zero and the second map is restriction. \square

Corollary 1.6 *Let C be a rational tree and let $R \subset C$ be a connected union of irreducible components of C . Let \mathcal{F} be a locally free sheaf on C such that the restriction of \mathcal{F} to each irreducible component of C which is not in R is generated by global sections. Then $h^1(C, \mathcal{F}) = h^1(R, \mathcal{F}|_R)$.*

Proof. Proceed by induction on the number ℓ of irreducible components of C not in R . If $\ell = 0$, there is nothing to prove. Suppose $\ell \geq 1$. Let C_1 be an end of C which is not an end of R , and let $p \in C_1$ be the node. The existence of such a component is easy to prove: since R is a proper subcurve of C , there must be a node of C where R meets a component not in R . Removing this node disconnects C into a connected component containing R and a connected component K disjoint from R . Clearly an end C_1 of the component K (different from the one meeting R if there is more than one end) is then also an end of C not contained in R . Since C_1 is a smooth rational curve, $\mathcal{F}|_{C_1} \simeq \bigoplus \mathcal{O}(a_j)$, with $a_j \geq 0$, thanks to the fact that $\mathcal{F}|_{C_1}$ is globally generated. In particular $h^1(C_1, \mathcal{F}(-p)) = 1$, and it is clear that we can now apply Lemma 1.5 to remove the component C_1 without changing h^1 and conclude using induction. \square

The last lemma of this section is an explicit computation of the cohomology of a locally free sheaf on a curve which will be extremely useful in the later sections.

Lemma 1.7 *Let C be a rational tree and $f : C \rightarrow S$ a morphism to a smooth surface. Let $p \in C$ be a node, denote by C_a and C_b the two irreducible components of C meeting at p . Let $\nu : \tilde{C} \rightarrow C$ be the normalization of C at p and let $\tilde{f} = f \circ \nu$. Suppose that:*

1. *the valences of the vertices C_a and C_b in the dual graph of C are at most 3, and*
2. *the map $f_* : \mathcal{T}_{C_a, p} + \mathcal{T}_{C_b, p} \longrightarrow \mathcal{T}_{S, f(p)}$ is surjective.*

Then

$$H^1(C, f^* \mathcal{T}_S) \cong H^1(\tilde{C}, \tilde{f}^* \mathcal{T}_S)$$

Proof. Consider the sequence on C

$$0 \longrightarrow f^* \mathcal{T}_S \longrightarrow \tilde{f}^* \mathcal{T}_S \xrightarrow{\epsilon} \mathcal{T}_{S, f(p)} \longrightarrow 0$$

Because $\mathcal{T}_{S, f(p)}$ is supported in dimension 0, $H^1(C, \mathcal{T}_{S, f(p)}) = 0$, and it is enough to prove that the sequence is exact on global sections. Let $\{p, q_a, r_a\}$

contain all the nodes of C on C_a and let $\{p, q_b, r_b\}$ contain the nodes on C_b . Consider now the following diagram:

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \mathcal{T}_{C_a}(-p - q_a - r_a) \oplus \mathcal{T}_{C_b}(-p - q_b - r_b) & & f^* \mathcal{T}_S \\
 \downarrow & & \downarrow \\
 \mathcal{T}_{C_a}(-q_a - r_a) \oplus \mathcal{T}_{C_b}(-q_b - r_b) & \longrightarrow & \tilde{f}^* \mathcal{T}_S \\
 \downarrow \alpha & & \downarrow \varepsilon \\
 \mathcal{T}_{C_a, p} \oplus \mathcal{T}_{C_b, p} & \xrightarrow{f_*} & \mathcal{T}_{S, f(p)} \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

where the unlabeled horizontal map is extension by zero. Since C_a and C_b are rational curves, their tangent bundles have degree 2 and α is surjective on global sections; f_* is surjective by assumption. It follows that ε is also surjective on global sections. \square

Remark. The second condition in the lemma is certainly satisfied if $f|_{C_a}$ and $f|_{C_b}$ are birational and the intersection of $f(C_a)$ and $f(C_b)$ is transverse at $f(p)$.

1.2 The Conormal Sheaf

Let $f : C \rightarrow X$ be a morphism from a connected, projective, at worst nodal curve C to a smooth projective variety X .

Definition 1.8 *The morphism $f : C \rightarrow X$ is called a stable map if C is a connected, projective, at worst nodal curve and every contracted component of geometric genus zero contains at least three singular points of C and every contracted component of geometric genus one contains at least one.*

We are interested in computing the obstruction space to deforming the stable map $f : C \rightarrow X$. Let $f^* \Omega_X^1 \rightarrow \Omega_C^1$ be the natural complex of sheaves associated with the differential of f and where the sheaf $f^* \Omega_X^1$ is in degree -1 and the sheaf Ω_C^1 is in degree 0. We know that the stability condition is equivalent to the vanishing of the group $\text{Hom}(f^* \Omega_X^1 \rightarrow \Omega_C^1, \mathcal{O}_C)$. The tangent space to $\overline{\mathcal{M}}_{0,0}(X, \beta)$ at f is given by the hypercohomology

group $\mathbb{E}xt^1(f^*\Omega_X^1 \rightarrow \Omega_C^1, \mathcal{O}_C)$. The obstruction space is a quotient of the hypercohomology group $\mathbb{E}xt^2(f^*\Omega_X^1 \rightarrow \Omega_C^1, \mathcal{O}_C)$. Denote by L_f^\bullet the complex $f^*\Omega_X^1 \rightarrow \Omega_C^1$, where the first sheaf is in degree -1 and the second one is in degree 0. Our strategy to compute these groups is to use the short exact sequence of complexes of sheaves:

$$\begin{array}{c}
 (0 \rightarrow 0) \\
 \downarrow \\
 (0 \rightarrow \Omega_C^1) \\
 \downarrow \\
 (f^*\Omega_X^1 \rightarrow \Omega_C^1) \\
 \downarrow \\
 (f^*\Omega_X^1 \rightarrow 0) \\
 \downarrow \\
 (0 \rightarrow 0)
 \end{array}$$

Applying the functor $\text{Hom}(-, \mathcal{O}_C)$ and using the long exact hypercohomology sequence we obtain:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}(L_f^\bullet, \mathcal{O}_C) & \longrightarrow & \text{Hom}(0 \rightarrow \Omega_C^1, \mathcal{O}_C) \\
 & & & & \\
 & \longrightarrow & \mathbb{E}xt^1(f^*\Omega_X^1 \rightarrow 0, \mathcal{O}_C) & \longrightarrow & \mathbb{E}xt^1(L_f^\bullet, \mathcal{O}_C) & \longrightarrow & \mathbb{E}xt^1(0 \rightarrow \Omega_C^1, \mathcal{O}_C) \\
 & & & & \\
 & \longrightarrow & \mathbb{E}xt^2(f^*\Omega_X^1 \rightarrow 0, \mathcal{O}_C) & \longrightarrow & \mathbb{E}xt^2(L_f^\bullet, \mathcal{O}_C) & \longrightarrow & \mathbb{E}xt^2(0 \rightarrow \Omega_C^1, \mathcal{O}_C)
 \end{array}$$

We can now rewrite many of these terms. First of all, the stability condition is equivalent to $\text{Hom}(L_f^\bullet, \mathcal{O}_C) = 0$. Also, remembering the fact that all the complexes are concentrated in degrees -1 and 0, and using the fact that $f^*\Omega_X^1$ is locally free, that its dual is $f^*\mathcal{T}_X$ and the isomorphisms $\mathbb{E}xt^i(f^*\Omega_X^1, \mathcal{O}_C) \simeq H^i(C, \mathcal{T}_X)$ we obtain the sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}(\Omega_C^1, \mathcal{O}_C) & \longrightarrow & H^0(C, f^*\mathcal{T}_X) & \longrightarrow & \mathbb{E}xt^1(L_f^\bullet, \mathcal{O}_C) & \longrightarrow & (1.2) \\
 & & & & \\
 & \longrightarrow & \mathbb{E}xt^1(\Omega_C^1, \mathcal{O}_C) & \longrightarrow & H^1(C, f^*\mathcal{T}_X) & \longrightarrow & \mathbb{E}xt^2(L_f^\bullet, \mathcal{O}_C) & \longrightarrow & 0
 \end{array}$$

In particular we see that if $H^1(C, f^*\mathcal{T}_X) = 0$, then the obstruction group $\mathbb{E}xt^2(L_f^\bullet, \mathcal{O}_C)$ vanishes as well, i.e. the map is unobstructed, the space of stable maps has the expected dimension at f and the point represented by f is smooth (for the stack).

If we consider the dual sequence of (1.2) and use Serre duality we obtain the sequence

$$\begin{aligned} 0 \longrightarrow (\mathbb{E}xt^2)^\vee &\longrightarrow H^0(C, f^*\Omega_X^1 \otimes \omega_C) \xrightarrow{\alpha} H^0(C, \Omega_C^1 \otimes \omega_C) \longrightarrow \\ &\longrightarrow (\mathbb{E}xt^1)^\vee \longrightarrow H^1(C, f^*\Omega_X^1 \otimes \omega_C) \longrightarrow H^1(C, \Omega_C^1 \otimes \omega_C) \longrightarrow 0 \end{aligned}$$

It is easy to convince oneself that the morphism α is the morphism induced by the differential map $df : f^*\Omega_X^1 \rightarrow \Omega_C$, by tensoring with the dualizing sheaf and taking global sections. Associated to f we may define the sheaves \mathcal{C}_f and \mathcal{Q}_f on C , by requiring the following sequence to be exact:

$$0 \longrightarrow \mathcal{C}_f \longrightarrow f^*\Omega_X^1 \xrightarrow{df} \Omega_C^1 \longrightarrow \mathcal{Q}_f \longrightarrow 0 \quad (1.3)$$

Since the dualizing sheaf ω_C is locally free, tensoring by ω_C is exact and taking global sections is left exact. From these remarks we deduce that

$$H^0(C, \mathcal{C}_f \otimes \omega_C) \simeq \mathbb{E}xt^2(L_f^\bullet, \mathcal{O}_C)^\vee$$

and we conclude that in order to compute the obstruction space of f , it is enough to compute the global sections of the sheaf $\mathcal{C}_f \otimes \omega_C$.

Definition 1.9 *The sheaf \mathcal{C}_f defined in (1.3) is the conormal sheaf of f .*

We drop the subscript f , when the morphism is clear from the context.

Definition 1.10 *A sheaf \mathcal{F} on a scheme of pure dimension one is pure if the support of every non-zero section has pure dimension one.*

It is clear that a locally free sheaf is pure. In fact, any subsheaf of a locally free sheaf is pure, and more generally any subsheaf of a pure sheaf is pure. In particular, the sheaves \mathcal{C}_f defined in (1.3) are pure.

Definition 1.11 *A point $p \in C$ is called a break for the morphism f (or simply a break), if the sheaf \mathcal{C}_f is not locally free at p . We say that the morphism f has no breaks if the sheaf \mathcal{C}_f is locally free.*

It is clear from the definition that a smooth point of C is never a break.

Definition 1.12 *Suppose the morphism f is finite. A point $p \in C$ is called a ramification point, if it belongs to the support of the sheaf \mathcal{Q}_f . We call ramification divisor of f the (Weil) divisor whose multiplicity at $p \in C$ is the length of \mathcal{Q} at p .*

Let $f_1 : C_1 \rightarrow X$ and $f_2 : C_2 \rightarrow X$ be non-constant morphisms from two smooth curves to a smooth surface. Suppose $p_1 \in C_1$ and $p_2 \in C_2$ are points such that $f_1(p_1) = f_2(p_2) = q$, let u and v be local coordinates on X near q and let x_1 and x_2 be local parameters for C_1 and C_2 near p_1 and p_2 respectively. Since f_1 and f_2 are not constant, there exist integers k_1 and k_2 such that

$$f_1^* : \begin{cases} u \mapsto x_1^{k_1} U_1(x_1) \\ v \mapsto x_1^{k_1} V_1(x_1) \end{cases} \quad f_2^* : \begin{cases} u \mapsto x_2^{k_2} U_2(x_2) \\ v \mapsto x_2^{k_2} V_2(x_2) \end{cases}$$

and $(U_1(0), V_1(0)), (U_2(0), V_2(0)) \neq (0, 0)$. We call a *tangent vector to C_i at p_i* any non-zero vector in $\mathcal{T}_q X$ proportional to $(U_i(0), V_i(0))$, and *tangent direction to C_i at p_i* the point in $\mathbb{P}(\mathcal{T}_q X)$ determined by a tangent vector to C_i at p_i . Geometrically, we may easily associate to each smooth point of $f_i(C_i)$ a tangent vector in the same way we did above, and then the tangent direction at any point is simply the limiting position of the tangent directions at the smooth points.

We say that C_1 and C_2 are *transverse at the point $q = f_i(p_i) \in X$* if their respective tangent directions at p_1 and p_2 are distinct and we will say that C_1 and C_2 are *not transverse at the point $q \in X$* if the tangent directions coincide.

Finally, we say that the morphism f_i is *ramified at p_i on C_i* if $k_i > 1$ and we say it is *unramified at C_i* if $k_i = 1$.

Lemma 1.13 *Let $f_i : C_i \rightarrow X$, $i \in \{1, 2\}$ be two non-constant morphisms from two smooth curves to a smooth surface X and let $p_1 \in C_1$ and $p_2 \in C_2$ be points such that $f_1(p_1) = f_2(p_2) = q$. Denote by \tilde{f}_1 and \tilde{f}_2 the morphisms induced by f_1 and f_2 from each curve to the blow-up of X at q , and assume $\tilde{f}_1(p_1) = \tilde{f}_2(p_2) = \tilde{q}$. Then the following conditions are equivalent:*

1. \tilde{f}_1 and \tilde{f}_2 are unramified at \tilde{q} and C_1 and C_2 are transverse at \tilde{q} ;
2. after possibly renumbering the curves C_1 and C_2 , there are coordinates u, v on X near q and x_i on C_i near p_i such that

$$\begin{aligned} f_1^* : \begin{cases} u \mapsto x_1 U_1(x_1) \\ v \mapsto x_1^3 V_1(x_1) \end{cases} & \quad U_1(0) \neq 0 \\ f_2^* : \begin{cases} u \mapsto x_2^m U_2(x_2) \\ v \mapsto x_2^{m+1} V_2(x_2) \end{cases} & \quad \begin{matrix} U_2(0), V_2(0) \neq 0 \\ m \geq 1 \end{matrix} \end{aligned} \tag{1.4}$$

Proof. Suppose we are given coordinates so that the f_i 's are given by (1.4). Let $b : \tilde{X} \rightarrow X$ be the blow-up morphism. Let $\tilde{u} := b^*u$, and note that near the point \tilde{q} the function \tilde{u} is a local equation for $E := b^{-1}(q)$, since the tangent vector to the curve locally defined by the vanishing of u is $(0, 1)$, while a tangent vector to the curve C_1 at q is $(1, 0)$. It follows that we may write $b^*v = \tilde{u} \cdot \tilde{v}$, and \tilde{u}, \tilde{v} is a local system of parameters on \tilde{X} at \tilde{q} such that b and its rational inverse b^{-1} are given by:

$$b^* : \begin{cases} u \mapsto \tilde{u} \\ v \mapsto \tilde{u}\tilde{v} \end{cases} \quad (b^*)^{-1} : \begin{cases} \tilde{u} \mapsto u \\ \tilde{v} \mapsto v/u \end{cases} \quad (1.5)$$

Thus the morphisms $\tilde{f}_i : C_i \rightarrow \tilde{X}$ are given by

$$\begin{aligned} \tilde{f}_1^* : \begin{cases} \tilde{u} \mapsto x_1 U_1(x_1) \\ \tilde{v} \mapsto x_1^2 \frac{V_1(x_1)}{U_1(x_1)} \end{cases} & U_1(0) \neq 0 \\ \tilde{f}_2^* : \begin{cases} \tilde{u} \mapsto x_2^m U_2(x_2) \\ \tilde{v} \mapsto x_2 \frac{V_2(x_2)}{U_2(x_2)} \end{cases} & U_2(0), V_2(0) \neq 0 \end{aligned}$$

Clearly these maps are unramified at $x_i = 0$ and since $(1, 0)$ and $(\star, 1)$ are tangent vectors at \tilde{q} to C_1 and C_2 respectively, the maps are also transverse at \tilde{q} . This simple computation proves the first half of the lemma.

Suppose conversely that in the blow-up \tilde{X} of X at q , the curves C_1 and C_2 meet transversely at the point $\tilde{q} = \tilde{f}_i(p_i) \in \tilde{X}$. Fix coordinates x_1 on C_1 at p_1 and x_2 on C_2 at p_2 , and choose coordinates u, v near q and \tilde{u}, \tilde{v} near \tilde{q} such that (1.5) are the equations of the blow-up morphism. We have

$$\tilde{f}_1^* : \begin{cases} \tilde{u} \mapsto x_1^{k_1} U_1(x_1) \\ \tilde{v} \mapsto x_1^{k_1} V_1(x_1) \end{cases} \quad \tilde{f}_2^* : \begin{cases} \tilde{u} \mapsto x_2^{k_2} U_2(x_2) \\ \tilde{v} \mapsto x_2^{k_2} V_2(x_2) \end{cases}$$

with $(U_1(0), V_1(0)), (U_2(0), V_2(0))$ linearly independent. By changing v to $v - \frac{V_1(0)}{U_1(0)}u$ and \tilde{v} to $\tilde{v} - \frac{V_1(0)}{U_1(0)}\tilde{u}$, we may assume that $V_1(0) = 0$, while preserving the equations of b . With these assumptions, $(1, 0)$ and $(\star, 1)$ are tangent vectors at \tilde{q} to C_1 and C_2 respectively. Moreover, since \tilde{f}_i is not ramified at p_i , necessarily $k_i = 1$. We have therefore

$$\begin{aligned} \tilde{f}_1^* : \begin{cases} \tilde{u} \mapsto x_1 U_1(x_1) \\ \tilde{v} \mapsto x_1^2 \bar{V}_1(x_1) \end{cases} & \quad f_1^* = \tilde{f}_1^* \circ b^* : \begin{cases} u \mapsto x_1 U_1(x_1) \\ v \mapsto x_1^3 U_1(x_1) \bar{V}_1(x_1) \end{cases} \\ \tilde{f}_2^* : \begin{cases} \tilde{u} \mapsto x_2 U_2(x_2) \\ \tilde{v} \mapsto x_2 V_2(x_2) \end{cases} & \quad f_2^* = \tilde{f}_2^* \circ b^* : \begin{cases} u \mapsto x_2 U_2(x_2) \\ v \mapsto x_2^2 U_2(x_2) V_2(x_2) \end{cases} \end{aligned}$$

where $U_1(0), V_2(0) \neq 0$.

In order to conclude we still need to show that $U_2(x_2)$ is not identically zero, but this is clear, since otherwise the morphism f_2 would be constant (i.e. the morphism \tilde{f}_2 would map C_2 to the exceptional divisor E). \square

Definition 1.14 *In the situation described by the previous lemma, the two curves C_1 and C_2 are simply tangent at q .*

We will see later (Lemma 1.16) that being simply tangent is closely related to the local structure of the conormal sheaf.

Lemma 1.15 *Suppose that X is a smooth surface and let $f : C \rightarrow X$ be a morphism from a curve C consisting of two irreducible components C_1 and C_2 , meeting in a node p . Denote by f_i the restriction of f to C_i and by $p_i \in C_i$ the point $p \in C$, and suppose that f does not contract any component of C and that C_1 and C_2 meet transversely at $f(p)$. Then there are the following cases:*

1. *Both maps f_1 and f_2 are unramified at p .*

Then \mathcal{C}_f is locally free and the following sequence is exact

$$0 \longrightarrow \mathcal{C}_f \longrightarrow \mathcal{C}_{f_1}(-p) \oplus \mathcal{C}_{f_2}(-p) \longrightarrow \mathcal{C}_{f,p} \longrightarrow 0$$

2. *f_i is unramified at p on C_i and f_{3-i} is ramified at p on C_{3-i} ($i \in \{1, 2\}$)*

Then \mathcal{C}_f is not locally free (i.e. p is a break point) and

$$\mathcal{C}_f \cong \mathcal{C}_{f_i}(-p) \oplus \mathcal{C}_{f_{3-i}}(-2p)$$

3. *Both maps f_1 and f_2 are ramified at p .*

Then \mathcal{C}_f is not locally free and

$$\mathcal{C}_f \cong \mathcal{C}_{f_1}(-p) \oplus \mathcal{C}_{f_2}(-p)$$

Proof. We can write

$$f^* : \begin{cases} u \longmapsto x^{k_1} U_1(x) + y^{k_2} U_2(y) \\ v \longmapsto x^{l_1} V_1(x) + y^{l_2} V_2(y) \end{cases}$$

where $l_1 > k_1$, $k_2 > l_2$ and $U_1(0), V_2(0) \neq 0$. We thus have

$$\begin{aligned} \mathcal{O}_{C,p} \cdot du + \mathcal{O}_{C,p} \cdot dv &\xrightarrow{df} \left(\mathcal{O}_{C,p} \cdot dx + \mathcal{O}_{C,p} \cdot dy \right) / (ydx + xdy) \\ du &\rightarrow x^{k_1-1} \left(k_1 U_1(x) + x U_1'(x) \right) dx + y^{k_2-1} \left(k_2 U_2(y) + y U_2'(y) \right) dy \\ dv &\rightarrow x^{l_1-1} \left(l_1 V_1(x) + x V_1'(x) \right) dx + y^{l_2-1} \left(l_2 V_2(y) + y V_2'(y) \right) dy \end{aligned}$$

In order to simplify this expression, let us define α_1 to be the invertible function $k_1 U_1(x) + x U_1'(x)$ and α_2 to be the invertible function $l_2 V_2(y) + y V_2'(y)$. Choosing $\frac{du}{\alpha_1}$ and $\frac{dv}{\alpha_2}$ as a basis for the $\mathcal{O}_{C,p}$ -module $f^* \Omega_{X,p}^1$ we may write

$$\frac{du}{\alpha_1} \longrightarrow x^{k_1-1} dx + y^{k_2-1} \varphi(y) dy$$

$$\frac{dv}{\alpha_2} \longrightarrow x^{l_1-1} \psi(x) dx + y^{l_2-1} dy$$

Note that

$$\begin{aligned} y^{k_2-1} \varphi(y) &= \frac{y^{k_2-1}}{k_1 U_1(0)} \left(k_2 U_2(y) + y U_2'(y) \right) \\ x^{l_1-1} \psi(x) &= \frac{x^{l_1-1}}{l_2 V_2(0)} \left(l_1 V_1(x) + x V_1'(x) \right) \end{aligned}$$

The elements of the kernel of df are determined by the condition

$$f_1(x, y) \frac{du}{\alpha_1} + f_2(x, y) \frac{dv}{\alpha_2} \longmapsto r(x, y) (ydx + xdy)$$

which translates to

$$\begin{aligned} x^{k_1-1} \left(f_1(x, y) + x^{l_1-k_1} f_2(x, y) \psi(x) \right) &= yr(x, y) = yr(0, y) \\ y^{l_2-1} \left(y^{k_2-l_2} f_1(x, y) \varphi(y) + f_2(x, y) \right) &= xr(x, y) = xr(x, 0) \end{aligned} \tag{1.6}$$

We are now going to split the three cases.

Case 1. In this case $k_1 = l_2 = 1$, and equation (1.6) becomes

$$f_1(x, y) + x^{l_1-1} f_2(x, y) \psi(x) = yr(x, y) = yr(0, y)$$

$$y^{k_2-1} f_1(x, y) \varphi(y) + f_2(x, y) = xr(x, y) = xr(x, 0)$$

This clearly implies that neither f_1 nor f_2 have constant term and hence we may write $f_1(x, y) = xg_1(x) + yh_1(y)$ and $f_2(x, y) = xg_2(x) + yh_2(y)$ and we have

$$\begin{aligned} yh_1(y) + x\left(g_1(x) + x^{l_1-1}g_2(x)\psi(x)\right) &= yr(x, y) \\ xg_2(x) + y\left(y^{k_2-1}h_1(y)\varphi(y) + h_2(y)\right) &= xr(x, y) \end{aligned}$$

Therefore

$$\begin{aligned} xg_1(x) &= -x^{l_1}g_2(x)\psi(x) \\ yh_2(y) &= -y^{k_2}h_1(y)\varphi(y) \\ yh_1(y) &= yr(x, y) \\ xg_2(x) &= xr(x, y) \end{aligned}$$

and near p all elements of the kernel of df are multiples of

$$\kappa := \left(-x^{l_1}\psi(x) + y\right)\frac{du}{\alpha_1} + \left(x - y^{k_2}\varphi(y)\right)\frac{dv}{\alpha_2}$$

It is very easy to check that κ is also in the kernel of df . This in particular implies that \mathcal{C}_f is locally free near p . The restriction of κ to C_1 (which is defined near p by $y = 0$) is

$$\frac{-x^{l_1}\psi(x)}{\alpha_1}du + \frac{x}{l_2V_2(0)}dv = \frac{x}{l_2V_2(0)}\left(\frac{-x^{l_1-1}(l_1V_1(x) + xV_1'(x))}{U_1(x) + xU_1'(x)}du + dv\right)$$

On the other hand, the restriction of f to C_1 is given by

$$f_1^* : \begin{cases} u \longmapsto xU_1(x) \\ v \longmapsto x^{l_1}V_1(x) \end{cases}$$

and the kernel of df_1 is clearly generated by

$$\frac{-x^{l_1-1}(l_1V_1(x) + xV_1'(x))}{U_1(x) + xU_1'(x)}du + dv$$

Thus $\mathcal{C}_{f_1}(-p)$ is generated near p by the same generator of $\mathcal{C}_f|_{C_1}$. Similarly, $\mathcal{C}_{f_2}(-p)$ and $\mathcal{C}_f|_{C_2}$ have the same generator near p . Hence 1 follows.

Case 2. Let us go back to equation (1.6) and substitute $k_1 = 1$ and $l_2 \geq 2$:

$$f_1(x, y) + x^{l_1-1} f_2(x, y) \psi(x) = yr(x, y) = yr(0, y)$$

$$y^{l_2-1} \left(y^{k_2-l_2} f_1(x, y) \varphi(y) + f_2(x, y) \right) = xr(x, y) = xr(x, 0)$$

This implies that $r(x, 0) = 0$, i.e. $r(x, y) = yr(y)$. Thus we have $f_1(x, y) = xg_1(x) + y^2r(y)$, and finally $f_2(x, y) = xg_2(x) - y^{k_2-l_2+2}\varphi(y)r(y)$. Substituting back, we find

$$xg_1(x) + x^{l_1}g_2(x)\psi(x) = 0$$

Therefore $xg_1(x) = -x^{l_1}g_2(x)\psi(x)$ and near p the kernel of df is generated by

$$x \left(-x^{l_1-1}\psi(x) \frac{du}{\alpha_1} + \frac{dv}{\alpha_2} \right) \quad \text{and} \quad y^2 \left(\frac{du}{\alpha_1} - y^{k_2-l_2}\varphi(y) \frac{dv}{\alpha_2} \right)$$

(as before, it is very easy to check that these elements lie indeed in the kernel of df).

We thus see that \mathcal{C}_f is not locally free near p ; since clearly the terms in brackets in the previous expression are local generators for \mathcal{C}_{f_1} and \mathcal{C}_{f_2} respectively near p , we deduce that $\mathcal{C}_f \cong \mathcal{C}_{f_1}(-p) \oplus \mathcal{C}_{f_2}(-2p)$. Thus 2 follows.

Case 3. Once more we refer to (1.6), now with $k_1, l_2 \geq 2$. In this case, the equations imply that $r(x, y) = 0$, and thus $f_1(x, y) = -x^{l_1-k_1}f_2(x, y)\psi(x) + yh_1(y)$. Substituting back in (1.6), we find

$$y^{l_2-1} \left(y^{k_2-l_2+1}h_1(y)\varphi(y) + f_2(x, y) \right) = 0$$

i.e. $f_2(x, y) = xg_2(x) - y^{k_2-l_2+1}h_1(y)\varphi(y)$ and therefore

$$f_1(x, y) = -x^{l_1-k_1+1}g_2(x)\psi(x) + yh_1(y)$$

By inspection we see that choosing $(g_2(x), h_1(y)) = (1, 0)$ or $(0, 1)$ yields elements of the kernel of df . Thus near p the kernel of df is generated by

$$x \left(-x^{l_1-k_1}\psi(x) \frac{du}{\alpha_1} + \frac{dv}{\alpha_2} \right) \quad \text{and} \quad y \left(\frac{du}{\alpha_1} + y^{k_2-l_2}\varphi(y) \frac{dv}{\alpha_2} \right)$$

We thus see again that \mathcal{C}_f is not locally free near p . Since clearly the terms in brackets in the previous expression are local generators for \mathcal{C}_{f_1} and \mathcal{C}_{f_2} respectively near p , it follows that $\mathcal{C}_f \cong \mathcal{C}_{f_1}(-p) \oplus \mathcal{C}_{f_1}(-p)$. Thus 3 is established, and with it the lemma. \square

Now that we have treated the transverse case, we need an analogous lemma for the non-transverse case.

Lemma 1.16 *Suppose that X is a smooth surface and let $f : C \rightarrow X$ be a morphism from a curve C consisting of two irreducible components C_1 and C_2 , meeting in a node p . Denote by f_i the restriction of f to C_i and let $p_i \in C_i$ be the point $p \in C$. Suppose that f does not contract any component of C and that C_1 and C_2 do not meet transversely at $f(p)$. Then there are the following cases:*

1. C_1 and C_2 are simply tangent at $f(p)$.

Then \mathcal{C}_f is not locally free and

$$\mathcal{C}_f \cong \mathcal{C}_{f_1}(-p) \oplus \mathcal{C}_{f_2}(-p)$$

2. C_1 and C_2 are not simply tangent at $f(p)$.

Then \mathcal{C}_f is locally free and the following sequence is exact

$$0 \longrightarrow \mathcal{C}_f \longrightarrow \mathcal{C}_{f_1} \oplus \mathcal{C}_{f_2} \longrightarrow \mathcal{C}_{f,p} \longrightarrow 0 \quad (1.7)$$

Proof. We proceed as before, and we can write

$$f^* : \begin{cases} u \mapsto x^{k_1}U_1(x) + y^{k_2}U_2(y) \\ v \mapsto x^{l_1}V_1(x) + y^{l_2}V_2(y) \end{cases}$$

where $l_1 > k_1$, $l_2 > k_2$ and $U_1(0), U_2(0) \neq 0$. By exchanging if necessary the roles of C_1 and C_2 , we may further assume that $k_1 \leq k_2$. Then we have

$$\begin{aligned} \mathcal{O}_{C,p} \cdot du + \mathcal{O}_{C,p} \cdot dv &\xrightarrow{df} \left(\mathcal{O}_{C,p} \cdot dx + \mathcal{O}_{C,p} \cdot dy \right) / (ydx + xdy) \\ du &\rightarrow x^{k_1-1} \left(k_1 U_1(x) + x U_1'(x) \right) dx + y^{k_2-1} \left(k_2 U_2(y) + y U_2'(y) \right) dy \\ dv &\rightarrow x^{l_1-1} \left(l_1 V_1(x) + x V_1'(x) \right) dx + y^{l_2-1} \left(l_2 V_2(y) + y V_2'(y) \right) dy \end{aligned}$$

Let $\alpha_1 := k_1 U_1(x) + x U_1'(x)$ and $\alpha_2 := l_2 V_2(y) + y V_2'(y)$. We may write

$$du \longrightarrow \alpha_1 x^{k_1-1} dx + \alpha_2 y^{k_2-1} dy$$

$$dv \longrightarrow x^{l_1-1} \psi(x) dx + y^{l_2-1} \varphi(y) dy$$

Note that $\alpha_1(0), \alpha_2(0) \neq 0$. The kernel of this morphism is determined by the condition

$$f_1(x, y)du + f_2(x, y)dv \longmapsto r(x, y)(ydx + xdy)$$

which translates to

$$f^* : \begin{cases} x^{k_1-1} \left(\alpha_1 f_1(x, y) + x^{l_1-k_1} \psi(x) f_2(x, y) \right) = yr(x, y) = yr(0, y) \\ y^{k_2-1} \left(\alpha_2 f_1(x, y) + y^{l_2-k_2} \varphi(y) f_2(x, y) \right) = xr(x, y) = xr(x, 0) \end{cases} \quad (1.8)$$

Let us now consider separately some cases.

Case 1. $k_1 = 1$ and $l_2 = k_2 + 1$. (i.e. f is not ramified on C_1 and $f(C_1)$ and $f(C_2)$ are simply tangent). We know we may also assume $l_1 \geq 3$. Equations (1.8) imply (multiplying the second one by y if $k_2 = 1$)

$$\begin{aligned} \alpha_1 f_1(x, y) + x^{l_1-1} \psi(x) f_2(x, y) &= yr(x, y) \\ y \left(\alpha_2 f_1(x, y) + y \varphi(y) f_2(x, y) \right) &= 0 \end{aligned}$$

From the second equation we deduce that $f_1(x, y) = xg(x) - \frac{y\varphi(y)}{\alpha_2} f_2(x, y)$, and substituting in the first equation we find

$$xg(x) = -\frac{x^{l_1-1} \psi(x)}{\alpha_1} f_2(x, y) + \frac{y\varphi(y)}{\alpha_2} f_2(x, y) + \frac{yr(x, y)}{\alpha_1}$$

This gives us the equations

$$\begin{aligned} xg(x) &= -\frac{x^{l_1-1} \psi(x)}{\alpha_1} f_2(x, y) \\ \frac{y\varphi(y)}{\alpha_2} f_2(x, y) &= -\frac{yr(x, y)}{\alpha_1} \end{aligned}$$

It follows that $f_2(x, y) = xh(x) - \frac{\alpha_2}{\alpha_1 \varphi(y)} r(x, y)$. Observe now that choosing $h(x) = 0$ and $r(x, y) = 1$ gives the element $\left(\frac{x^{l_1-1} \alpha_2 \psi(x)}{\alpha_1^2 \varphi(y)} + \frac{y}{\alpha_1} \right) du - \frac{\alpha_2}{\alpha_1 \varphi(y)} dv$ whose image under df is ydx (remember we are assuming $l_1 \geq 3$), which is not zero. Therefore, $r(x, y)$ (and hence f_2) cannot have a constant term, which implies that all elements of the kernel are combinations of

$$x \left(-\frac{x^{l_1-1} \psi(x)}{\alpha_1} du + dv \right) \quad \text{and} \quad y \left(-\frac{y\varphi(y)}{\alpha_2} du + dv \right)$$

Clearly these elements are also in the kernel of df and the terms in the brackets are local generators for \mathcal{C}_{f_1} and \mathcal{C}_{f_2} . Thus $\mathcal{C}_f \cong \mathcal{C}_{f_1}(-p) \oplus \mathcal{C}_{f_2}(-p)$.

Case 2a. $k_1 = 1, l_2 \geq k_2 + 2$. Equations (1.8) imply

$$\begin{aligned}\alpha_1 f_1(x, y) + x^{l_1-1} \psi(x) f_2(x, y) &= yr(x, y) \\ y(\alpha_2 f_1(x, y) + y^{l_2-k_2} \varphi(y) f_2(x, y)) &= 0\end{aligned}$$

From the second equation we deduce that

$$f_1(x, y) = xg(x) - \frac{y^{l_2-k_2} \varphi(y)}{\alpha_2} f_2(x, y)$$

and substituting in the first equation we find

$$xg(x) = -\frac{x^{l_1-1} \psi(x)}{\alpha_1} f_2(x, y) + \frac{y^{l_2-k_2} \varphi(y)}{\alpha_2} f_2(x, y) + \frac{yr(x, y)}{\alpha_1}$$

This gives us the equations

$$\begin{aligned}xg(x) &= -\frac{x^{l_1-1} \psi(x)}{\alpha_1} f_2(x, y) \\ \frac{y^{l_2-k_2} \varphi(y)}{\alpha_2} f_2(x, y) &= -\frac{yr(x, y)}{\alpha_1}\end{aligned}$$

Therefore all elements of the kernel are multiples of

$$-\left(\frac{x^{l_1-1} \psi(x)}{\alpha_1} + \frac{y^{l_2-k_2} \varphi(y)}{\alpha_2}\right) du + dv$$

By inspection these elements are also in the kernel of df and the restrictions $-\frac{x^{l_1-1} \psi(x)}{\alpha_1} du + dv$ and $-\frac{y^{l_2-k_2} \varphi(y)}{\alpha_2} du + dv$ are generators for \mathcal{C}_{f_1} and \mathcal{C}_{f_2} . In particular there is a short exact sequence as in (1.7).

Case 2b. $k_1, k_2 \geq 2$. Then (1.8) implies $r(x, y) = 0$ and from the first equation we may write $f_1(x, y) = -\frac{x^{l_1-k_1} \psi(x)}{\alpha_1} f_2(x, y) + yh(y)$ and substituting in the second equation we obtain

$$y^{k_2-1} \left(y\alpha_2 h(y) + y^{l_2-k_2} \varphi(y) f_2(x, y) \right) = 0 \implies yh(y) = -\frac{y^{l_2-k_2} \varphi(y)}{\alpha_2} f_2(x, y)$$

Therefore near p any element of the kernel of df is a multiple of

$$-\left(\frac{x^{l_1-k_1} \psi(x)}{\alpha_1} + \frac{y^{l_2-k_2} \varphi(y)}{\alpha_2}\right) du + dv$$

and it is easy to check that this element lies indeed in the kernel of df . Thus \mathcal{C}_f is locally free and since $-\frac{x^{l_1-k_1}\psi(x)}{\alpha_1}du + dv$ and $-\frac{y^{l_2-k_2}\varphi(y)}{\alpha_2}du + dv$ are the local generators for \mathcal{C}_{f_1} and \mathcal{C}_{f_2} , we deduce that we have a short exact sequence as in (1.7). This completes the proof of the lemma. \square

Let $f : C \rightarrow X$ be a morphism from a connected, projective, nodal curve of arithmetic genus zero to a smooth surface X . In view of the two previous lemmas, we partition the set of nodes of C in five disjoint sets:

τ_{uu} is the set of nodes p such that the two components of C meeting at p are transverse at $f(p)$ and both are unramified;

τ_{ur} is the set of nodes p such that the two components of C meeting at p are transverse at $f(p)$ and one is unramified and the other one is ramified;

τ_{rr} is the set of nodes p such that the two components of C meeting at p are transverse at $f(p)$ and both are ramified;

ν_2 is the set of nodes p such that the two components of C meeting at p are simply tangent at $f(p)$;

ν_l is the set of nodes p such that the two components of C meeting at p are not transverse and not simply tangent at $f(p)$.

Thus it follows from the lemmas that the sheaf \mathcal{C}_f is locally free at the nodes τ_{uu} and ν_l , while it is not free at the others. Let C_1, \dots, C_ℓ be the components of C . Then we let τ_{uu}^i denote the divisor on C_i of nodes lying in τ_{uu} , and similarly for the other types of nodes. Note that only one of the definitions above is not symmetric, namely τ_{ur} (and τ_{ru}^i). To take care of this, let us introduce one more divisor on each component of C : let τ_{ru}^i be the divisor on C_i consisting of all nodes p of C on C_i , such that the two components of C through p are transverse at $f(p)$, and the restriction of f to these two components is ramified only on C_i .

Often we will denote by the same symbol a divisor on a curve and its degree. For instance we write equations like

$$\begin{aligned} \sum_i (\tau_{uu}^i + \tau_{ur}^i + \tau_{rr}^i + \nu_2^i + \nu_l^i) &= 2\#\{\text{nodes of } C\} \\ \sum_i (\tau_{uu}^i + \tau_{ur}^i + \tau_{ru}^i + \tau_{rr}^i + \nu_2^i + \nu_l^i) &= 2\#\{\text{nodes of } C\} + \tau_{ur} \end{aligned}$$

Given a coherent sheaf \mathcal{F} on a curve C , let $\tau(\mathcal{F})$ denote the subsheaf generated by the sections whose support has dimension at most 0 and let \mathcal{F}^{free} be the sheaf $\mathcal{F}/\tau(\mathcal{F})$. By definition the sheaf \mathcal{F}^{free} is pure.

Proposition 1.17 *Let $f : C \rightarrow X$ be a stable map of genus zero with no contracted components to a smooth surface X , with canonical divisor K_X . Let C_1, \dots, C_ℓ be the irreducible components of C . Then we have*

$$\begin{aligned} \deg \left((\mathcal{C}_f \otimes \omega_C) |_{C_i}^{free} \right) &= f_*[C_i] \cdot K_X - \deg \tau_{ru}^i + \deg \nu_l^i + \deg \mathcal{Q}_i \\ \chi(\mathcal{C}_f \otimes \omega_C) &= f_*[C] \cdot K_X + \deg \tau_{rr} + \deg \nu_2 + \\ &\quad + 2 \deg \nu_l + \sum \deg \mathcal{Q}_i + 1 \end{aligned} \quad (1.9)$$

Moreover, let $\nu : \tilde{C} \rightarrow C$ be the normalization of C at the nodes in $\tau_{ur} \cup \tau_{rr} \cup \nu_2$. Then, the sheaf \mathcal{C}_f is the pushforward of a locally free sheaf on \tilde{C} .

Proof. This is simply a matter of collecting the information we already proved in the previous lemmas. Thanks to Lemma 1.15 and Lemma 1.16 we have the following short exact sequence of sheaves on C

$$0 \longrightarrow \mathcal{C}_f \longrightarrow \oplus_i \mathcal{C}_{f_i} (-\tau_{uu}^i - \tau_{ur}^i - 2\tau_{ru}^i - \tau_{rr}^i - \nu_2^i) \longrightarrow \mathcal{C}_f|_{\tau_{uu}} \oplus \mathcal{C}_f|_{\nu_l} \longrightarrow 0$$

Note that the sheaf in the middle on the component C_i is twisted down by all nodes of C on C_i , with the exception of the nodes in ν_l^i , which do not appear, and the nodes in τ_{ru}^i , which “appear twice.” Hence we can write the divisor by which we are twisting \mathcal{C}_{f_i} as $-val[C_i] - \tau_{ru}^i + \nu_l^i$ (we denote by $val[C_i]$ the valence of the vertex $[C_i]$ in the dual graph of C).

To compute the degree of the sheaf \mathcal{C}_{f_i} , remember that there is an exact sequence

$$0 \longrightarrow \mathcal{C}_{f_i} \longrightarrow f_i^* \Omega_X^1 \longrightarrow \Omega_{C_i}^1 \longrightarrow \mathcal{Q}_i \longrightarrow 0$$

Therefore we have $\deg \mathcal{C}_{f_i} = f_*[C_i] \cdot K_X + 2 + \deg \mathcal{Q}_i$. Thus, we may rewrite the previous sequence as follows

$$\begin{aligned} 0 \longrightarrow \mathcal{C}_f \longrightarrow \oplus_i \mathcal{O}_{C_i} \left(f_*[C_i] \cdot K_X + 2 - val[C_i] - \tau_{ru}^i + \nu_l^i + \deg \mathcal{Q}_i \right) \longrightarrow \\ \longrightarrow \mathcal{C}_f|_{\tau_{uu}} \oplus \mathcal{C}_f|_{\nu_l} \longrightarrow 0 \end{aligned}$$

The dualizing sheaf ω_C is invertible and on the component C_i has degree equal to $-2 + val[C_i]$. Thus twisting the previous sequence by ω_C we obtain

(using the isomorphisms $\mathcal{C}_f|_{\tau_{uu}} \otimes \omega_C \simeq \mathcal{C}_f|_{\tau_{uu}}$ and $\mathcal{C}_f|_{\nu_l} \otimes \omega_C \simeq \mathcal{C}_f|_{\nu_l}$)

$$\begin{aligned} 0 \longrightarrow \mathcal{C}_f \otimes \omega_C \longrightarrow \oplus_i \mathcal{O}_{C_i} \left(f_*[C_i] \cdot K_X - \tau_{ru}^i + \nu_l^i + \deg \mathcal{Q}_i \right) \longrightarrow \\ \longrightarrow \mathcal{C}_f|_{\tau_{uu}} \oplus \mathcal{C}_f|_{\nu_l} \longrightarrow 0 \end{aligned} \quad (1.10)$$

The first identity in (1.9) follows. For the second one, note that $\sum \tau_{ru}^i = \tau_{ur}$ and $\sum \nu_l^i = 2\nu_l$ and compute Euler characteristics of (1.10):

$$\begin{aligned} \chi(\mathcal{C}_f \otimes \omega_C) &= \sum_i (f_*[C_i] \cdot K_X - \tau_{ru}^i + \nu_l^i + \deg \mathcal{Q}_i + 1) - \deg \tau_{uu} - \deg \nu_l = \\ &= f_*[C] \cdot K_X - \tau_{ur} + 2\nu_l + \sum_i \deg \mathcal{Q}_i + \\ &\quad + \#\{\text{components of } C\} - \tau_{uu} - \nu_l \end{aligned}$$

Remember that the dual graph of f is a tree and hence $\#\{\text{components}\} = \#\{\text{nodes of } C\} + 1 = \tau_{uu} + \tau_{ur} + \tau_{rr} + \nu_2 + \nu_l + 1$. Using this, we conclude

$$\chi(\mathcal{C}_f \otimes \omega_C) = f_*[C] \cdot K_X + \tau_{rr} + \nu_2 + 2\nu_l + 1 + \sum_i \deg \mathcal{Q}_i$$

and the proposition is proved. \square

The next proposition has a similar proof, but deals with morphisms with contracted components. As for the previous case, it is useful to introduce two more subsets of the nodes on contracted components, depending on the behaviour of $f : \bar{C} \rightarrow X$ near the node. We let

ρ_u be the set of nodes p such that f is constant on one of the two components, and it is unramified on the other;

ρ_r be the set of nodes p such that f is constant on one of the two components, and it is ramified on the other.

Proposition 1.18 *Let $f : \bar{C} \rightarrow X$ be a stable map of genus zero to a smooth surface X , with canonical divisor K_X . Let $\bar{C} = C \cup R$, where $C = C_1 \cup \dots \cup C_\ell$ is the union of all components of \bar{C} which are not contracted by f , and R is the union of all components of \bar{C} contracted by f . Let r be the number of connected components of the curve R (equivalently, $r = \chi(\mathcal{O}_R)$). Then we have*

$$\begin{aligned} \deg \left((\mathcal{C}_f \otimes \omega_{\bar{C}}) |_{C_i}^{free} \right) &= f_*[C_i] \cdot K_X + \mathcal{Q}_i - \tau_{ru}^i + \nu_l^i + \rho_u^i + \rho_r^i \quad (1.11) \\ \chi(\mathcal{C}_f \otimes \omega_{\bar{C}}) &= f_*[C] \cdot K_X + \sum \mathcal{Q}_i + 1 + \tau_{rr} + \nu_2 + 2\nu_l + \\ &\quad + \rho_u + 2\rho_r - 3r \end{aligned}$$

Proof. For the first formula in (1.11), we only need to check the local behaviour of \mathcal{C}_f near a node between C_i and a contracted component R_j . As before, let x be a local coordinate on C_i near the node p between C_i and R_j and let y be a local coordinate on R_j near p . Let u, v be local coordinates on X near $f(p)$ and suppose that the tangent direction to the vanishing set of u near $f(p)$ is the tangent direction to C_i at $f(p)$. We have

$$f^* : \begin{cases} u & \longmapsto x^k U(x) \\ v & \longmapsto x^{k+1} V(x) \end{cases} \quad U(0) \neq 0$$

for some $k \geq 1$. The sheaf \mathcal{C}_f near p is the kernel of the map

$$df : \mathcal{O}_{\bar{C},p} \cdot du + \mathcal{O}_{\bar{C},p} \cdot dv \longrightarrow \left(\mathcal{O}_{\bar{C},p} \cdot dx + \mathcal{O}_{\bar{C},p} \cdot dy \right) / (ydx + xdy)$$

$$du \longrightarrow x^{k-1} \left(kU(x) + xU'(x) \right) dx$$

$$dv \longrightarrow x^k \left((k+1)V(x) + xV'(x) \right) dx$$

It is readily seen that

$$x \left((k+1)V(x) + xV'(x) \right) du - \left(kU(x) + xU'(x) \right) dv$$

is a local generator for the kernel of df . Note that this means that we may pretend that the component R_j is not there for the purpose of computing the contribution of the node p , regardless of whether $f|_{C_i}$ ramifies or not at p . This is enough to prove the first formula in (1.11).

To prove the second one, we carry the previous analysis slightly further, and note that the image of df contains the torsion section ydx if and only if

f does not ramify at p . Remember that we have the diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow \tau & & \\
 & & 0 & & \downarrow & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{C}_f & \longrightarrow & f^*\Omega_X^1 & \longrightarrow & \Omega_{\bar{C}}^1 \longrightarrow \mathcal{Q}_{\bar{C}} \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \bigoplus_{C' \subset \bar{C}} (f|_{C'})^* \Omega_X^1 & & \bigoplus_{C' \subset \bar{C}} \Omega_{C'}^1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \bigoplus_{\nu \in \text{Sing}(\bar{C})} \Omega_{X,\nu}^1 & & 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

where C' ranges over the irreducible components of \bar{C} and τ denotes the torsion subsheaf of $\Omega_{\bar{C}}^1$. We deduce that

$$\chi(\mathcal{C}_f \otimes \omega_{\bar{C}}) = \chi(f^*\Omega_X^1 \otimes \omega_{\bar{C}}) - \chi(\Omega_{\bar{C}}^1 \otimes \omega_{\bar{C}}) + \chi(\mathcal{Q}_{\bar{C}} \otimes \omega_{\bar{C}})$$

and we know that

$$\begin{aligned}
 \chi(f^*\Omega_X^1 \otimes \omega_{\bar{C}}) &= f_*[\bar{C}] \cdot K_X - 4\#\{\text{components of } \bar{C}\} + 2 \sum_{C' \subset \bar{C}} \text{val}(C') + \\
 &\quad + 2\#\{\text{components of } \bar{C}\} - 2\#\{\text{nodes of } \bar{C}\} = \\
 &= f_*[\bar{C}] \cdot K_X - 2
 \end{aligned}$$

$$\chi(\Omega_{\bar{C}}^1 \otimes \omega_{\bar{C}}) = \#\{\text{nodes of } \bar{C}\} - 3\#\{\text{components of } \bar{C}\} + \sum_{C' \subset \bar{C}} \text{val}(C') = -3$$

$$\chi(\mathcal{Q}_{\bar{C}} \otimes \omega_{\bar{C}}) = \chi(\mathcal{Q}_C \otimes \omega_{\bar{C}}) + \chi(\Omega_R^1 \otimes \omega_{\bar{C}}) + \rho_r$$

where \mathcal{Q}_C is the cokernel of the differential of the restriction of f to the union C of the non-contracted components. By what we saw above, the sheaf \mathcal{Q}_C behaves like when there are no contracted components. The Euler

characteristic of $\Omega_R^1 \otimes \omega_{\bar{C}}$ is given by

$$\begin{aligned} \chi(\Omega_R^1 \otimes \omega_{\bar{C}}) &= \#\{\text{nodes of } \bar{R}\} - 3\#\{\text{irreducible components of } \bar{R}\} + \\ &\quad + \sum_{R' \subset R} \text{val}(R') = -3\#\{\text{connected components of } \bar{R}\} = \\ &= -3r \end{aligned}$$

We collect all these numbers as we did before and conclude. \square

1.3 Deformations of Stable Maps

1.4 Dimension Estimates

In what follows we refer to the integer $-C \cdot K_X$ as the anticanonical degree (or simply as the degree) of a curve C in X , where K_X is the canonical divisor of X .

We consistently use the following notational convention: if $f : \bar{C} \rightarrow X$ is a morphism and \bar{C}_1 denotes a component of \bar{C} , we denote the image of \bar{C}_1 by C_1 , and in general, a symbol with a bar over it denotes an object on the source curve \bar{C} , while the same symbol without the bar over it denotes the image of the object in X .

Definition 1.19 ([Ko] II.3.6). *Let $f, g \in \text{Hom}(\bar{C}, X)$; we say g is a deformation of f , if there is an irreducible subscheme of $\text{Hom}(\bar{C}, X)$ containing f and g . We say that a general deformation of f has some property if there is an open subset $U \subset \text{Hom}(\bar{C}, X)$ containing f and a dense open subset $V \subset U$ such that all $f' \in V$ have that property.*

When we choose a general deformation g of a morphism f , we assume that g is a deformation of f , i.e. that f and g lie in the same irreducible component of $\text{Hom}(\bar{C}, X)$.

Lemma 1.20 *Let $f : \mathbb{P}^1 \rightarrow X$ be a free morphism; then $-f(\mathbb{P}^1) \cdot K_X \geq 2$. If moreover f is birational onto its image, then a general deformation of f is free and it is an immersion.*

Proof. Since f is free, $f^*\mathcal{T}_X$ is globally generated, and hence the normal sheaf \mathcal{N}_f is also. Thus we have

$$0 \leq \deg \mathcal{N}_f = \deg f^*\mathcal{T}_X - 2 = -f(\mathbb{P}^1) \cdot K_X - 2$$

For the second assertion, by [Ko] Complement II.3.14.4, a general deformation of f is of the form $f_t : \mathbb{P}^1 \xrightarrow{g_t} \mathbb{P}^1 \xrightarrow{h_t} X$, where h_t is an immersion. Since it is also true that a general deformation of a birational map is still birational, we see that for a general deformation f_t of f , g_t is an isomorphism, and f_t is an immersion. Clearly being free is also an open property. \square

Fix a free rational curve $\beta \subset X$ and let $d = -\beta \cdot K_X$.

Definition 1.21 Denote by $\overline{\mathcal{M}}_{bir}(X, \beta)$ the closure in $\overline{\mathcal{M}}_{0,0}(X, \beta)$ of the set of free morphisms $f : \mathbb{P}^1 \rightarrow X$ such that f is birational onto its image.

We want to prove that given $r \leq d - 1$ general points $p_1, \dots, p_r \in X$, in all irreducible components of $\overline{\mathcal{M}}_{bir}(X, \beta)$ there is an f whose image contains all the p_i 's.

Proposition 1.22 Let $f : \mathbb{P}^1 \rightarrow X$ be an immersion, and let d be the degree of the image of f . Let c_1, \dots, c_r be distinct points where f is an embedding. The natural morphism

$$F^{(r)} : (\mathbb{P}^1)^r \times \text{Hom}(\mathbb{P}^1, X) \longrightarrow X^r$$

$$(d_1, \dots, d_r; [g]) \longmapsto (g(d_1), \dots, g(d_r))$$

is smooth at the point $(c_1, \dots, c_r; [f])$ if and only if $r \leq d - 1$.

Proof. Recall the sequence defining \mathcal{N}_f :

$$0 \longrightarrow \mathcal{T}\mathbb{P}^1 \xrightarrow{df^t} f^*\mathcal{T}X \longrightarrow \mathcal{N}_f \longrightarrow 0 \quad (1.12)$$

Let us prove first of all that $\text{Hom}(\mathbb{P}^1, X)$ is smooth at $[f]$. From (1.12), it follows that $\deg \mathcal{N}_f = -f_*(\mathbb{P}^1) \cdot K_X - 2 = d - 2 \geq -1$, and since f is an immersion, \mathcal{N}_f is locally free. Thus from the long exact sequence associated to (1.12) we deduce that $H^1(\mathbb{P}^1, f^*\mathcal{T}X) = 0$, and by [Ko] Theorem II.1.2 it follows that $\text{Hom}(\mathbb{P}^1, X)$ is smooth at $[f]$.

Consider now the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \oplus \mathcal{T}_{c_i} \mathbb{P}^1 & \longrightarrow & \mathcal{T}_{(\mathbb{C}; [f])} (\mathbb{P}^1)^r \times \text{Hom}(\mathbb{P}^1, X) & \longrightarrow & \mathcal{T}_{[f]} \text{Hom}(\mathbb{P}^1, X) \longrightarrow 0 \\ & & \downarrow \sim & & \downarrow dF^{(r)} & & \downarrow \delta \\ 0 & \longrightarrow & \oplus \mathcal{T}_{f(c_i)} f(\mathbb{P}^1) & \longrightarrow & \oplus \mathcal{T}_{f(c_i)} X & \longrightarrow & \oplus \mathcal{N}_{f, c_i} \longrightarrow 0 \end{array}$$

The top row is clear, since we have the isomorphism

$$\mathcal{T}_{(\underline{c};[f])}(\mathbb{P}^1)^r \times \mathrm{Hom}(\mathbb{P}^1, X) \simeq \oplus \mathcal{T}_{c_i} \mathbb{P}^1 \oplus \mathcal{T}_{[f]} \mathrm{Hom}(\mathbb{P}^1, X)$$

For the second row, restrict the sequence (1.12) to $\{c_1, \dots, c_r\}$ and note that f induces an isomorphism $\mathcal{T}_{c_i} \mathbb{P}^1 \simeq \mathcal{T}_{f(c_i)} f(\mathbb{P}^1)$, since f is an embedding at the c_i . The first vertical arrow is induced by f , while δ is the quotient map, followed by the evaluation map ([Ko] Proposition II.3.5):

$$\begin{array}{ccc} \mathcal{T}_{[f]} \mathrm{Hom}(\mathbb{P}^1, X) \simeq H^0(\mathbb{P}^1, f^* \mathcal{T}_X) & \xrightarrow{q} & H^0(\mathbb{P}^1, \mathcal{N}_f) \\ & \searrow \delta & \downarrow ev \\ & & \oplus \mathcal{N}_{f, c_i} \end{array}$$

The morphism q is induced by the long exact sequence associated to (1.12), and the next term in the sequence is $H^1(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}) = 0$. Therefore q is surjective. Observe that $dF^{(r)}$ is surjective if and only if δ is surjective, and finally, δ is surjective if and only if the evaluation map ev is surjective. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{N}_f(-c_1 - \dots - c_r) \longrightarrow \mathcal{N}_f \longrightarrow \oplus \mathcal{N}_{f, c_i} \longrightarrow 0 \quad (1.13)$$

Remember that $\deg \mathcal{N}_f = d - 2$, and since f is an immersion, $\mathcal{N}_f \simeq \mathcal{O}_{\mathbb{P}^1}(d - 2)$. Thus $H^1(\mathbb{P}^1, \mathcal{N}_f) = 0$, and the sequence on global sections induced by (1.13) is exact if and only if $H^1(\mathbb{P}^1, \mathcal{N}_f(-c_1 - \dots - c_r)) = 0$, i.e. if and only if $\deg \mathcal{N}_f(-c_1 - \dots - c_r) = d - 2 - r \geq -1$. Therefore $H^0(\mathbb{P}^1, \mathcal{N}_f) \rightarrow \oplus \mathcal{N}_{f, c_i}$ is surjective if and only if $r \leq d - 1$, and hence $dF^{(r)}$ is surjective if and only if $r \leq d - 1$. \square

Let $f : \mathbb{P}^1 \rightarrow X$ be an immersion representing an element of $\overline{\mathcal{M}}_{bir}(X, \beta)$, and denote by ${}_f \overline{\mathcal{M}}_{bir}(X, \beta)$ the irreducible component of $\overline{\mathcal{M}}_{bir}(X, \beta)$ containing f . Denote by $\mathcal{H}^f \subset \mathrm{Hom}(\mathbb{P}^1, X)$ the irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X)$ containing $[f]$ (remember that $\mathrm{Hom}(\mathbb{P}^1, X)$ is smooth at $[f]$).

There is an action

$$\begin{aligned} \mathrm{Aut}(\mathbb{P}^1) \times (\mathbb{P}^1)^r \times \mathrm{Hom}(\mathbb{P}^1, X) &\longrightarrow (\mathbb{P}^1)^r \times \mathrm{Hom}(\mathbb{P}^1, X) \\ (\varphi, (c_1, \dots, c_r; [g])) &\longmapsto (\varphi(c_1), \dots, \varphi(c_r); [g \circ \varphi^{-1}]) \end{aligned}$$

which clearly preserves the irreducible components of $\mathrm{Hom}(\mathbb{P}^1, X)$. Since f is not constant, the action of $\mathrm{Aut}(\mathbb{P}^1)$ has finite stabilizers.

Consider the diagram

$$\begin{array}{ccc} & (\mathbb{P}^1)^r \times \mathcal{H}^f & \\ \swarrow F^{(r)} & & \searrow M \\ X^r & & {}_f\overline{\mathcal{M}}_{bir}(X, \beta) \end{array}$$

where M is the projection onto the factor \mathcal{H}^f followed by the natural map that quotients out the action of $\text{Aut}(\mathbb{P}^1)$.

Let us compute the dimensions of some of these spaces. The morphism M is obviously dominant, while Proposition 1.22 (together with Lemma 1.20) implies that $F^{(r)}$ is dominant if $r \leq d-1$. Thus we may compute

$$\dim({}_f\overline{\mathcal{M}}_{bir}(X, \beta)) = \dim((\mathbb{P}^1)^r \times \mathcal{H}) - r - 3 = -f(\mathbb{P}^1) \cdot K_X - 1 = d - 1$$

Let $c_1, \dots, c_r \in \mathbb{P}^1$ be $r \leq d-1$ distinct points where f is an isomorphism onto its image and let $p_i = f(c_i)$.

Let $p := (c_1, \dots, c_r; [f]) \in (\mathbb{P}^1)^r \times \text{Hom}(\mathbb{P}^1, X)$; it follows from Proposition 1.22 that

$$\dim(F^{(r)})^{-1}(p_1, \dots, p_r) = r + \dim \mathcal{H}^f - 2r = -f(\mathbb{P}^1) \cdot K_X + 2 - r = d - r + 2$$

Denote by $\overline{\mathcal{M}}_{bir}(X, \beta)(p_1, \dots, p_r)$ the (closure of the) image under M of $(F^{(r)})^{-1}(p_1, \dots, p_r)$, alternatively

$$\overline{\mathcal{M}}_{bir}(X, \beta)(p_1, \dots, p_r) := \left\{ [f] \in \overline{\mathcal{M}}_{bir}(X, \beta) \mid \text{Image}(f) \supset \{p_1, \dots, p_r\} \right\}$$

Since $\text{Aut}(\mathbb{P}^1)$ acts with finite stabilizers on $(F^{(r)})^{-1}(p_1, \dots, p_r)$, we may compute

$$\dim \overline{\mathcal{M}}_{bir}(X, \beta)(p_1, \dots, p_r) = d - r - 1 \quad (1.14)$$

1.5 Independent Points

The next lemma analyzes the case of curves through $d-1$ general points.

Lemma 1.23 *For a general $(d-1)$ -tuple (p_1, \dots, p_{d-1}) of points of X^{d-1} , all the morphisms in $\overline{\mathcal{M}}_{bir}(X, \beta)(p_1, \dots, p_{d-1})$ are immersions.*

Proof. Let $\mathcal{I} \subset (\mathbb{P}^1)^{d-1} \times \mathcal{H}^f$ be the set of all d -tuples $(c_1, \dots, c_{d-1}; [g])$ such that g is not an immersion; Lemma 1.20 implies that \mathcal{I} is a proper

closed subset of $(\mathbb{P}^1)^{d-1} \times \mathcal{H}^f$. Note that \mathcal{I} is $\text{Aut}(\mathbb{P}^1)$ -invariant. Consider the morphism $F^{(d-1)}$. By Proposition 1.22 and Lemma 1.20 this morphism is dominant, hence the general fiber of this morphism has dimension $d-1-f(\mathbb{P}^1) \cdot K_X + 2 - 2(d-1) = d+2-d+1 = 3$, thus the fibers of this morphism are $\text{Aut}(\mathbb{P}^1)$ -orbits, since they are stable under the action of $\text{Aut}(\mathbb{P}^1)$. If the general fiber of $F^{(d-1)}$ met \mathcal{I} , then we would have

$$\dim \mathcal{I} \geq 2(d-1) + 3 = 2d+1 = (d-1) + (d+2) = \dim \left((\mathbb{P}^1)^{d-1} \times \mathcal{H}^f \right)$$

and \mathcal{I} would equal $(\mathbb{P}^1)^{d-1} \times \mathcal{H}^f$, which contradicts Lemma 1.20. Thus there is an open dense subset \mathcal{U} in X^{d-1} not meeting the image of \mathcal{I} . For any $(d-1)$ -tuple $(p_1, \dots, p_{d-1}) \in \mathcal{U}$ we have that

$$\overline{\mathcal{M}}_{bir}(X, \beta)(p_1, \dots, p_{d-1}) := M \left((F^{(d-1)})^{-1}(p_1, \dots, p_{d-1}) \right) \subset \overline{\mathcal{M}}_{bir}(X, \beta)$$

consists only of (finitely many) immersions. \square

We now want to prove that for a general choice of $d-2$ points on X , all the resulting morphisms in $\overline{\mathcal{M}}_{bir}(X, \beta)$ through them have reduced image. To achieve this, let us first introduce the following notion.

Definition 1.24 *We say that r points p_1, \dots, p_r in X are independent if the following conditions hold:*

1. *no k of them are contained in a rational curve of degree k ;*
2. *the normalization of a rational curve of degree k in X through $k-1$ of them is an immersion.*

Proposition 1.22, Lemma 1.23 and the dimension estimates (1.14) easily imply that for any $r \geq 1$ there are r -tuples of independent points if there are free rational curves of anticanonical degree $d \geq r+1$, and that there are rational curves of anticanonical degree d through r independent points if $d \geq r+1$.

We are ready to prove the following result.

Lemma 1.25 *Let $C \subset X$ be a divisor of anticanonical degree $d \geq 3$ such that each reduced irreducible component is rational. Let $p_1, \dots, p_{d-2} \in C$ be a $(d-2)$ -tuple of independent points. The divisor C has at most two irreducible components and it is reduced.*

Proof. Denote by C_1, \dots, C_ℓ the reduced irreducible components of C . For each curve C_i let d_i be the degree of C_i , m_i be the multiplicity of C_i in C and δ_i be the number of points p_1, \dots, p_{d-2} lying on C_i . Then we have $\sum m_i d_i = d$ and $\delta_i \leq d_i - 1$. Therefore

$$d - 2 = \sum \delta_i \leq \sum d_i - \ell \leq \sum m_i d_i - \ell = d - \ell$$

Thus $\ell \leq 2$, and if $\ell = 2$, then all inequalities are equalities and hence $m_1 = m_2 = 1$. If $\ell = 1$, then C_1 is a rational curve of degree d_1 on X containing $d - 2$ independent points. It follows that $d_1 \geq d - 1$ and $m_1 d_1 = d$ and hence $d \geq m_1(d - 1)$, or $(m_1 - 1)d \leq m_1$. Since $d \geq 3$ this implies $d_1 = d$ and $m_1 = 1$. \square

Lemma 1.26 *Let $p_1, \dots, p_r \in X$ be $r \geq 2$ independent points, and let $\alpha \subset X$ be an integral curve of degree $r + 2$ of geometric genus zero containing p_1, \dots, p_r . Let B be a smooth connected projective curve and let $F : B \rightarrow \overline{\mathcal{M}}_{bir}^\alpha(p_1, \dots, p_r)$ be a non-constant morphism. The reducible curves in the family parametrized by B cannot always contain a component mapped isomorphically to a curve of anticanonical degree strictly smaller than two.*

Proof. Consider the following fiber product diagram

$$\begin{array}{ccc} S & \xrightarrow{\quad} & \overline{\mathcal{M}}_{0,1}(X, \alpha) \\ \downarrow & & \downarrow \\ B & \xrightarrow{\quad} & \overline{\mathcal{M}}_{bir}^\alpha(p_1, \dots, p_r) \hookrightarrow \overline{\mathcal{M}}_{0,0}(X, \alpha) \end{array}$$

thus $S \rightarrow B$ is the pull-back of the universal family.

It follows that $S \rightarrow B$ is a surface whose general fiber over B is a smooth rational curve and with a finite number of fibers consisting of exactly two smooth rational curves (Lemma 1.25) meeting transversely at a point, corresponding to the reducible curves in the family B . By hypothesis $S \rightarrow B$ admits r contractible sections. Suppose that in all reducible fibers of S one component is mapped to a curve of anticanonical degree strictly smaller than two. Denote the components in S mapped to such curves by L_1, \dots, L_t , and the other components in the respective fiber by Q_1, \dots, Q_t (thus $L_i + Q_i$ represents the numerical class of a fiber, for all i 's). By definition of independent points, the sections of $S \rightarrow B$ cannot meet the components L_i . Since $L_i \subset S$ is a smooth rational curve of self-intersection $L_i^2 = L_i \cdot (Q_i + L_i) - L_i \cdot Q_i = -1$, we may contract all the L_i to obtain a smooth surface $S' \rightarrow B$, which is a \mathbb{P}^1 -bundle over the curve B . Since the contracted

curves did not meet the r sections, there still are $r \geq 2$ negative sections of $S' \rightarrow B$, but there can be at most one negative section in a \mathbb{P}^1 -bundle. Thus there must be reducible fibers in the family B all of whose components are mapped to curves of anticanonical degree at least two. \square

Lemma 1.27 *Let $f : \mathbb{P}^1 \rightarrow X$ be a non-constant morphism to a smooth surface X and suppose that $f^*\mathcal{T}_X$ is globally generated. Denote by \mathcal{M}_f the irreducible component of $\overline{\mathcal{M}}_{0,0}(X, f_*[\mathbb{P}^1])$ containing $[f]$ and by $C \subset X$ the integral curve $f(\mathbb{P}^1)$. Let $\mathcal{M}_{f,C}$ be the locus of stable maps*

$$\mathcal{M}_{f,C} := \left\{ [g] \in \mathcal{M}_f \mid \text{image}(g) = C \right\}$$

Then we have

$$\text{codim}(\mathcal{M}_{f,C}, \mathcal{M}_f) \leq 1$$

Equality holds if and only if $f_[\mathbb{P}^1] = \delta C$ for some positive integer δ and $K_X \cdot C = -2$.*

Proof. Using [Ko] Proposition II.3.7, we may deform f so that the image of the resulting morphism avoids a point on C . It follows that $\mathcal{M}_{f,C} \subsetneq \mathcal{M}_f$, and hence, $\mathcal{M}_{f,C}$ being closed, that it has codimension at least one.

To prove the second assertion, note that any morphism $\phi : R \rightarrow X$ from a rational tree with image contained in C is such that $\phi^*\mathcal{T}_X$ is globally generated. This is obvious on each irreducible component of R : the morphism factors through the normalization of C and a multiple cover, and under the normalization the pull-back of \mathcal{T}_X is globally generated. Thus $\phi^*\mathcal{T}_X$ is globally generated on each component of R , and hence it is globally generated on R .

Let Γ be the dual graph of some morphism in \mathcal{M}_f . Let \mathcal{M}_f^Γ be the subscheme of \mathcal{M}_f consisting of morphisms with dual graph Γ ; then

$$\text{codim}(\mathcal{M}_{f,C} \cap \mathcal{M}_f^\Gamma, \mathcal{M}_f^\Gamma) \geq 1 \tag{1.15}$$

Indeed, let n be the number of vertices of Γ and consider the scheme $\widetilde{\mathcal{M}}_f^\Gamma$:

$$\left\{ [g : K \rightarrow X; p_1, \dots, p_n] \in \mathcal{M}_{0,n}(X, f_*[\mathbb{P}^1]) \mid \begin{array}{l} [g] \in \mathcal{M}_f^\Gamma \text{ and the points} \\ p_1, \dots, p_n \text{ lie in different} \\ \text{components of } K \end{array} \right\}$$

Clearly there is a surjective morphism $\widetilde{\mathcal{M}}_f^\Gamma \rightarrow \mathcal{M}_f^\Gamma$, and let

$$\widetilde{\mathcal{M}}_{f,C}^\Gamma := (\mathcal{M}_{f,C} \cap \mathcal{M}_f^\Gamma) \times_{\mathcal{M}_f^\Gamma} \widetilde{\mathcal{M}}_f^\Gamma$$

Let $g : K \rightarrow X$ represent a morphism in $\widetilde{\mathcal{M}}_{f,C}^\Gamma$; again by [Ko] Proposition II.3.7 we may deform g to miss a point of C , while still lying in $\widetilde{\mathcal{M}}_f^\Gamma$ and thus (1.15) follows.

Suppose that $\text{codim}(\mathcal{M}_{f,C}, \mathcal{M}_f) = 1$. It is clear that $f_*[\mathbb{P}^1] = \delta C$ for some positive integer δ .

Using (1.15) it follows that the general morphism in every component of maximal dimension of $\mathcal{M}_{f,C}$ has irreducible domain, and hence these components of $\mathcal{M}_{f,C}$ are dominated by $\mathcal{M}_{0,0}(\mathbb{P}^1, \delta)$, where the morphisms are induced by composition with the normalization map $\nu : \mathbb{P}^1 \rightarrow C$. We have $\dim \mathcal{M}_{f,C} \leq \dim \mathcal{M}_{0,0}(\mathbb{P}^1, \delta) = 2\delta - 2$, and also $\dim \mathcal{M}_f = (-K_X \cdot C)\delta - 1$. We already know (Lemma 1.20) that $-K_X \cdot C \geq 2$, and hence we must have $-K_X \cdot C = 2$ and $\dim \mathcal{M}_{f,C} = 2\delta - 2$. \square

1.6 Sliding moves

The next lemma and its corollary allow us to construct irreducible subschemes in the boundary of the spaces $\overline{\mathcal{M}}_{0,0}(X, \beta)$. First, let us introduce some notation that will be used in the lemma.

Let $f : \bar{C} \rightarrow X$ be a stable map of genus zero to the smooth surface X . Let \bar{C}_0 be a connected subcurve, let $\bar{C}_1, \dots, \bar{C}_\ell$ be the connected components of the closure of $\bar{C} \setminus \bar{C}_0$. Let \bar{C}_{0i} be the irreducible component of \bar{C}_0 meeting \bar{C}_i , and let $\bar{C}_{i,1}$ be the irreducible component of \bar{C}_i meeting \bar{C}_0 and let the intersection point of \bar{C}_{0i} and $\bar{C}_{i,1}$ be \bar{p}_i . Denote by f_i the restriction of f to \bar{C}_i , for $i \in \{0, \dots, \ell\}$.

Let $V \subset \overline{\mathcal{M}}_{0,\ell}(X, f_*[\mathbb{P}^1]) \times (\bar{C}_1 \times \dots \times \bar{C}_\ell)$ be the subscheme consisting of all points $([g; \bar{c}_1, \dots, \bar{c}_\ell]; \bar{c}'_1, \dots, \bar{c}'_\ell)$, such that $g(\bar{c}_i) = f(\bar{c}'_i)$ and $[g; \bar{c}_1, \dots, \bar{c}_\ell]$ is in the same irreducible component of $\overline{\mathcal{M}}_{0,\ell}(X, f_*[\mathbb{P}^1])$ as $[f; \bar{p}_1, \dots, \bar{p}_\ell]$.

Lemma 1.28 *With notation as above, assume also that a general deformation of f_0 is generated by global sections, \bar{C}_{0i} is not contracted by f and $f(\bar{C}_{0i}) \not\supset f(\bar{C}_{i,1})$, for all i 's. It follows that every irreducible component of V containing $([f_0; \bar{p}_1, \dots, \bar{p}_\ell]; \bar{p}_1, \dots, \bar{p}_\ell)$ surjects onto the irreducible component of $\overline{\mathcal{M}}_{0,0}(X, f_*[\mathbb{P}^1])$ containing $[f]$.*

Proof. Let Φ be an irreducible component of $\overline{\mathcal{M}}_{0,0}(X, f_*[\mathbb{P}^1])$ containing (the stable reduction of) $[f]$. Define \mathcal{C} by the Cartesian square on the left

and \underline{ev} as the composite of the maps in the diagram

$$\begin{array}{ccccc}
 & & \underline{ev} := (ev_1, \dots, ev_\ell) & & \\
 & \nearrow & & \searrow & \\
 \mathcal{C} & \xrightarrow{\quad} & \overline{\mathcal{M}}_{0,\ell}(X, f_*[\mathbb{P}^1]) & \longrightarrow & X^\ell \\
 \downarrow & & \downarrow & & \\
 \Phi & \xrightarrow{\quad} & \overline{\mathcal{M}}_{0,0}(X, f_*[\mathbb{P}^1]) & &
 \end{array}$$

Clearly, V is then defined by the diagram

$$\begin{array}{ccc}
 V & \xrightarrow{\quad} & (\bar{C}_1 \times \dots \times \bar{C}_\ell) \\
 \downarrow \iota & & \downarrow (f_1, \dots, f_\ell) \\
 \pi \swarrow & \mathcal{C} \xrightarrow{\quad \underline{ev} \quad} & X^\ell \\
 & \downarrow & \\
 & \Phi &
 \end{array}$$

and we have

$$V \subset W := \mathcal{C} \times (\bar{C}_1 \times \dots \times \bar{C}_\ell) \xrightarrow{P} \mathcal{C}$$

Obviously P is flat and since $\mathcal{C} \rightarrow \Phi$ is flat, it follows that $W \rightarrow \Phi$ is flat. The fiber of π at the point $[g]$ is given by

$$\pi^{-1}([g]) = \left\{ ([\tilde{g}; \bar{c}_1, \dots, \bar{c}_\ell]; \bar{c}'_1, \dots, \bar{c}'_\ell) \mid \tilde{g}(\bar{c}_i) = f_i(\bar{c}'_i) \right\}$$

where the stable reduction of \tilde{g} is g . If g has irreducible domain, and if the image of g does not contain any singular point of (the reduced scheme) $f(\bar{C}_1 \cup \dots \cup \bar{C}_\ell)$, nor does it contain any component of $f(\bar{C}_i)$, then the scheme $\pi^{-1}([g])$ is finite. Thanks to [Ko] Theorem II.7.6 and Proposition II.3.7, a general deformation g of f_0 satisfies the previous conditions; thus the general fiber of π in a neighbourhood of $[f]$ is finite and hence, letting $v_0 := ([f_0; \bar{p}_1, \dots, \bar{p}_\ell]; \bar{p}_1, \dots, \bar{p}_\ell)$, we conclude that $\dim_{v_0} V = \dim \Phi = \dim \mathcal{C} - \ell$.

Let $\kappa_i \in \mathcal{O}_{X, f(\bar{p}_i)}$ be a local equation of $f_i(\mathbb{P}^1)$; clearly the ℓ equations $P^*ev_1^*(\kappa_1), \dots, P^*ev_\ell^*(\kappa_\ell)$ define V near v_0 . Since $\dim V = \dim \mathcal{C} - \ell$, it follows that \mathcal{O}_{V, v_0} is a Cohen-Macaulay \mathcal{O}_{W, v_0} -module. Using [EGA4] Proposition 6.1.5, we deduce that \mathcal{O}_{V, v_0} is a flat $\mathcal{O}_{\Phi, [f_0]}$ -module, and the result follows. \square

Construction. Suppose $f : \bar{C} \rightarrow X$ is a stable map, and suppose $\bar{C} = \bar{C}_0 \cup \dots \cup \bar{C}_\ell$, where \bar{C}_i is a connected union of components for all i 's, such that $H^1(\bar{C}_0, f^* \mathcal{T}_X|_{\bar{C}_0}) = 0$ and all the irreducible components of \bar{C}_0 meeting \bar{C}_i are not contracted by f and the image of the component of \bar{C}_0 meeting \bar{C}_i does not contain the image of the corresponding component of \bar{C}_i for all i 's (this is the same condition required in Lemma 1.28).

We construct an irreducible subscheme $\mathrm{Sl}_f(\bar{C}_0)$ of $\overline{\mathcal{M}}_{0,0}(X, f_*[\bar{C}])$, consisting of morphisms $g : \bar{C}' \rightarrow X$ with the following properties:

- there is a decomposition $\bar{C}' = \bar{C}'_0 \cup \dots \cup \bar{C}'_\ell$, where \bar{C}'_i is a connected subcurve;
- there are isomorphisms $g|_{\bar{C}'_i} \simeq f|_{\bar{C}_i}$;
- there is a morphism $res : \mathrm{Sl}_f(\bar{C}_0) \rightarrow \overline{\mathcal{M}}_{0,0}(X, f_*[\bar{C}_0])$, which is surjective on the irreducible component containing $f|_{\bar{C}_0}$;
- there are morphisms $a_i : \mathrm{Sl}_f(\bar{C}_0) \rightarrow \bar{C}_i$, for $i \in \{1, \dots, \ell\}$.

Let $\bar{p}_i \in \bar{C}_0$ be the node between \bar{C}_0 and \bar{C}_i and $f_i := f|_{\bar{C}_i}$; by Lemma 1.28 we may find an irreducible subscheme $V \subset \overline{\mathcal{M}}_{0,\ell}(X, f_*[\bar{C}_0]) \times_{X^\ell} (\bar{C}_1 \times \dots \times \bar{C}_\ell)$ and a morphism $V \rightarrow \overline{\mathcal{M}}_{0,0}(X, f_*[\bar{C}_0])$ which is surjective onto the irreducible component containing f_0 .

Identify \bar{C}_i with $\overline{\mathcal{M}}_{0,1}(\bar{C}_i, [\bar{C}_i])$; thus we may write

$$V \subset \overline{\mathcal{M}}_{0,\ell}(X, f_*[\bar{C}_0]) \times_{X^\ell} \left(\overline{\mathcal{M}}_{0,1}(\bar{C}_1, [\bar{C}_1]) \times \dots \times \overline{\mathcal{M}}_{0,1}(\bar{C}_\ell, [\bar{C}_\ell]) \right)$$

Let $M_i \subset \bar{C}_0 \times P$ be the closed subscheme with closed points of the form $(\bar{c}_{0i}; [g; \bar{c}_{01}, \dots, \bar{c}_{0\ell}]; \bar{c}_1, \dots, \bar{c}_\ell)$. Let $N_i \subset \bar{C}_i \times P'$ be the closed subscheme with closed points of the form $(\bar{c}_i; [g; \bar{c}_{01}, \dots, \bar{c}_{0\ell}]; \bar{c}_1, \dots, \bar{c}_\ell)$. It is clear that projection onto the P' factor induces isomorphisms $M_i \simeq P'$ and $N_i \simeq P'$, and that $M_i \cap M_j = \emptyset$ for all $i \neq j$.

Construct the scheme $\bar{\mathcal{C}}$: glue to $\bar{C}_0 \times P'$ the schemes $\bar{C}_i \times P'$ along the subschemes $M_i \simeq N_i$, where the isomorphisms are the ones induced by projection onto the factor P' . By construction, there is a morphism $\bar{\mathcal{C}} \rightarrow P'$, whose fiber over the point $\bar{c} = ([g; \bar{c}_{01}, \dots, \bar{c}_{0\ell}]; \bar{c}_1, \dots, \bar{c}_\ell)$ is the curve $\bar{\mathcal{C}}_{\bar{c}}$ obtained by the nodal union of \bar{C}_0 and \bar{C}_i , for all i 's, where the nodes of $\bar{\mathcal{C}}_{\bar{c}}$ are at the points $\bar{c}_{0i} \in \bar{C}_0$ and $\bar{c}_i \in \bar{C}_{i,1} \subset \bar{C}_i$.

The morphism $\bar{\mathcal{C}} \rightarrow P'$ is flat on all irreducible components of P' (remember that P' is smooth) thanks to Theorem III.9.9 of [Ha], since all fibers $\bar{\mathcal{C}}_{\bar{c}}$ have geometric genus zero. Thus $\bar{\mathcal{C}} \rightarrow P'$ is a family of connected nodal projective curves of arithmetic genus zero.

A typical application of this construction can be found in the proof of Theorem 2.12 as well as in many of the later proofs.

2 Divisors of Small Degree: the Picard Lattice

2.1 The Nef Cone

We collect here some results on the nef cone of a del Pezzo surface. We prove a “numerical” decomposition of any nef divisor on a del Pezzo surface in Corollary 2.5. In the later sections we will show how to realize geometrically this decomposition.

Definition 2.1 *Let X_δ be a del Pezzo surface of degree $9 - \delta$. Suppose that $X_\delta \neq \mathbb{P}^1 \times \mathbb{P}^1$. We call an integral basis $\{\ell, e_1, \dots, e_\delta\}$ of $\text{Pic}(X_\delta)$ a standard basis if there is a presentation $b : X_\delta \rightarrow \mathbb{P}^2$ of X_δ as the blow up of \mathbb{P}^2 at δ points such that ℓ is the pull-back of the class of a line and the e_i ’s are the exceptional divisors of b .*

Lemma 2.2 *Let $C \subset X$ be an integral curve of canonical degree -1 on the smooth surface X . Then C^2 is odd and it is at least -1 .*

Proof. This is immediate from the adjunction formula:

$$C^2 + K_X \cdot C = 2p_a(C) - 2 \quad \implies \quad C^2 = 2p_a(C) - 1 \geq -1$$

The lemma is proved. \square

Lemma 2.3 *Let $C \subset X$ be a curve of canonical degree -1 on a del Pezzo surface of degree d . Either C is a (-1) -curve, or $d = 1$ and the divisor class of C is $-K_X$.*

Proof. Note that since $-K_X$ is ample, a curve of canonical degree -1 must be integral. If $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$, all divisor classes on X have even canonical degree, thus we may exclude this case. Let $\rho := C^2$ and $\delta = 9 - d$; by the previous lemma we know that $\rho \geq -1$ and it is odd. Moreover, if $\rho = -1$ then C is a (-1) -curve; suppose therefore that $\rho \geq 1$. By [Ma] Proposition IV.25.1 we may find a standard basis $\{\ell, e_1, \dots, e_\delta\}$ of the Picard group of X . If we write $C = a\ell - b_1e_1 - \dots - b_\delta e_\delta$, we have

$$\begin{cases} 3a - \sum_{i=1}^{\delta} b_i = 1 \\ a^2 - \sum_{i=1}^{\delta} b_i^2 = \rho \end{cases}$$

and these equations are easily seen to be equivalent to the following:

$$\left\{ \begin{array}{l} 3a - \sum_{i=1}^8 b_i = 1 \\ \sum_{i=1}^8 (a - 2b_i - 1)^2 = 4(1 - \rho) \\ b_i = 0 \quad i \geq \delta + 1 \end{array} \right.$$

We deduce that $\rho \leq 1$, and hence $\rho = 1$. We conclude that $a - 2b_i - 1 = 0$ for all i 's and hence $(a; b_1, \dots, b_8) = (2b + 1; b, \dots, b)$ and $3a - \sum b_i = 1$. Therefore $b = 1$, $\delta = 8$ and the divisor class of C is $(3\ell - e_1 - \dots - e_8) = -K_X$. \square

We need a criterion to determine which classes are nef on any del Pezzo surface X . This is immediate in the cases of del Pezzo surfaces of degree 9 and 8. If the degree is 9, then X is isomorphic to \mathbb{P}^2 . The non-negative multiples of the class of a line are the only nef divisors, and the only ample divisors are the positive such multiples. If the degree of the del Pezzo surface is 8, then there are two cases: either X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or X is isomorphic to the blow-up of \mathbb{P}^2 at one point.

If $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then any divisor class C on X is of the form $a_1 F_1 + a_2 F_2$, where F_1 and F_2 are the two divisor classes of $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$ and a_1 and a_2 are integers. Then C is nef if and only if $a_1, a_2 \geq 0$, while C is ample if and only if $a_1, a_2 > 0$.

If $X \simeq Bl_p(\mathbb{P}^2)$, then any divisor class C on X is of the form $a\ell - be$, where ℓ is the pull-back of the divisor class of a line in \mathbb{P}^2 , while e is the exceptional divisor. The divisor class C is nef if and only if $a \geq b \geq 0$, while C is ample if and only if $a > b > 0$.

The remaining cases are dealt with in the next Proposition.

Proposition 2.4 *Let X be a del Pezzo surface of degree $d \leq 7$. A divisor class $C \in \text{Pic}(X)$ is nef (respectively ample) if and only if $C \cdot L \geq 0$ (respectively $C \cdot L > 0$) for all (-1) -curves $L \subset X$.*

Proof. The necessity of the conditions is obvious. To establish the sufficiency, we only need to prove the result for nef classes, since the ample classes are precisely the ones in the interior of the nef cone. Proceed by induction on $r := 9 - d$.

If $r = 2$ write $C = a\ell - b_1 e_1 - b_2 e_2$, in some standard basis $\{\ell, e_1, e_2\}$. By assumption we know that $b_i \geq 0$ and $a \geq b_1 + b_2$. Thus we can write

$$C = (a - b_1 - b_2)\ell + b_1(\ell - e_1) + b_2(\ell - e_2)$$

which shows that C is a non-negative combination of nef classes.

Suppose $r > 2$. Let $n := \min\{C \cdot L ; L \subset X \text{ is a } (-1)\text{-curve}\}$; by assumption we know that $n \geq 0$. Let $\tilde{C} := C + nK_X$; for any (-1) -curve $L \subset X$ we have $\tilde{C} \cdot L = C \cdot L - n \geq 0$, and there is a (-1) -curve L' such that $\tilde{C} \cdot L' = 0$, by the definition of n .

Let $b : X \rightarrow X'$ be the contraction of the curve L' and note that X' is a del Pezzo surface of degree $9 - (r - 1)$. We have $\tilde{C} = b^*b_*\tilde{C} - rL'$ and

$$0 = \tilde{C} \cdot L' = b^*b_*\tilde{C} \cdot L' - rL' \cdot L' = b_*\tilde{C} \cdot b_*L' + r = r$$

and therefore $\tilde{C} = b^*b_*\tilde{C}$ is the pull-back of the divisor class $C' := b_*\tilde{C}$ on X' . Since all (-1) -curves on X' are images of (-1) -curves on X , by induction we know that C' is nef, and thus \tilde{C} is nef. Hence $C = \tilde{C} + n(-K_X)$ is a non-negative linear combination of nef divisors, and thus C is nef. \square

From this Proposition we deduce immediately the following Corollary.

Corollary 2.5 *Let X_δ be a del Pezzo surface of degree $9 - \delta \leq 8$. Let $D \in \text{Pic}(X_\delta)$ be a nef divisor. Then we can find*

- *non-negative integers n_2, \dots, n_δ ;*
- *a sequence of contraction of (-1) -curves*

$$X_\delta \longrightarrow X_{\delta-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 ;$$

- *a nef divisor $D' \in \text{Pic}(X_1)$;*

such that

$$D = n_\delta(-K_{X_\delta}) + n_{\delta-1}(-K_{X_{\delta-1}}) + \dots + n_2(-K_{X_2}) + D'$$

Proof. We proceed by induction on δ . If $\delta \leq 1$, there is nothing to prove.

Suppose that $\delta \geq 2$ and let $n := \min\{L \cdot D \mid L \subset X \text{ a } (-1)\text{-curve}\}$. By assumption we have $n \geq 0$. Let $\bar{D} := D + nK_{X_\delta}$; for every (-1) -curve $L \subset X_\delta$ we have

$$\bar{D} \cdot L = D \cdot L + nK_{X_\delta} \cdot L \geq n - n = 0$$

Thus thanks to the previous Proposition, \bar{D} is nef. By construction there is a (-1) -curve $L_0 \subset X$ such that $\bar{D} \cdot L_0 = 0$. Thus \bar{D} is the pull-back of a nef divisor on the del Pezzo surface $X_{\delta-1}$ obtained by contracting L_0 . By induction, we have a sequence of contractions

$$X_{\delta-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 ,$$

non-negative integers $n_2, \dots, n_{\delta-1}$ and a nef divisor D' on X_1 such that we may write $\bar{D} = n_{\delta-1}(-K_{X_{\delta-1}}) + \dots + n_2(-K_{X_2}) + D'$. Let $n_\delta := n$; with this notation we have

$$D = n_\delta(-K_{X_\delta}) + \bar{D}' = n_\delta(-K_{X_\delta}) + \dots + n_2(-K_{X_2}) + D'$$

and a sequence of contractions as in the statement of the corollary. This concludes the proof. \square

2.2 First Cases of the Main Theorem

Theorem 2.6 *Let X_δ be a del Pezzo surface of degree $9 - \delta$; then the linear system $|-K_{X_\delta}|$ has dimension $9 - \delta$. If $\delta = 8$, then $|-K_{X_8}|$ has a unique base-point; if $\delta \leq 7$, then $|-K_{X_\delta}|$ is base-point free and if $\delta \leq 6$ it is very ample.*

Proof. This result is well-known (cf. [Ma]). \square

Proposition 2.7 *Let X_δ be a del Pezzo surface of degree $9 - \delta \geq 3$. The scheme $\overline{\mathcal{M}}_{bir}(X_\delta, -K_{X_\delta})$ is birational to a $\mathbb{P}^{6-\delta}$ -bundle over X_δ ; in particular, it is rational and irreducible.*

Proof. The surface X_δ is embedded in $\mathbb{P}^{9-\delta}$ by the linear system $|-K_{X_\delta}|$. A general point $[f : \mathbb{P}^1 \rightarrow X_\delta]$ of $\overline{\mathcal{M}}_{bir}(X_\delta, -K_{X_\delta})$ corresponds to a morphism whose image has a unique singular point $p \in X_\delta$ and is uniquely determined by the hyperplane containing $f(\mathbb{P}^1)$. Such a hyperplane is tangent to X_δ at p . The hyperplanes in $\mathbb{P}^{9-\delta}$ intersecting X_δ in a curve with a singular point at p are precisely the hyperplanes containing the tangent plane to X_δ at p .

We thus have a rational morphism

$$\pi : \overline{\mathcal{M}}_{bir}(X_\delta, -K_{X_\delta}) \dashrightarrow X_\delta$$

assigning to $[f : \mathbb{P}^1 \rightarrow X_\delta]$ the unique singular point of $f(\mathbb{P}^1)$. The general point of the fiber of π over a general point $p \in X_\delta$ corresponds to a hyperplane containing the tangent plane to X_δ at p . The space of such hyperplanes is isomorphic to $\mathbb{P}^{6-\delta}$. Since X_δ is irreducible and the general fiber of π is also, it follows that $\overline{\mathcal{M}}_{bir}(X_\delta, -K_{X_\delta})$ is irreducible. \square

Remark. The schemes $\overline{\mathcal{M}}_{0,0}(X_\delta, -K_X)$ are not irreducible if X_δ is the blow-up of \mathbb{P}^2 at $\delta = 1$ or 2 points. Indeed, let X_1 be the blow-up of \mathbb{P}^2 at one

point p ; there are two morphisms

$$\begin{array}{ccc} & X_1 \subset \mathbb{P}^2 \times \mathbb{P}^1 & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}^2 & & \mathbb{P}^1 \end{array}$$

and the divisor class of a fiber of π_2 is $\ell - e$, where ℓ is the pull-back of the class of a line in \mathbb{P}^2 under π_1 and e is the exceptional fiber of π_1 . It is clear that the space of morphisms from a curve with dual graph

$$\bar{C}_1 \text{ --- } \bar{C}_2$$

where \bar{C}_1 is a (rational) triple cover of a fiber of π_2 and \bar{C}_2 is a double cover of the exceptional fiber of π_1 has dimension at least 7: there are 4 parameters for the triple cover of $\ell - e$, 1 for the choice of fiber of π_2 and 2 for the double cover of e .

Similarly, let X_2 be the blow-up of \mathbb{P}^2 at two distinct points p, q and let $\{\ell, e_1, e_2\}$ be a standard basis. It is clear that the space of morphisms from a curve with dual graph

$$\bar{C}_1 \text{ --- } \bar{D} \text{ --- } \bar{C}_2$$

where \bar{C}_i is a double cover of the (-1) -curve with divisor class e_i and \bar{D} is a triple cover of the (-1) -curve with divisor class $\ell - e_1 - e_2$ has dimension at least 8.

In both these cases it is easy to check (Proposition 1.17) that in fact the dimension of the components described is precisely the indicated lower bound.

It is also possible to show that these are the only irreducible components of $\overline{\mathcal{M}}_{0,0}(X, -K_X)$ besides the closure of $\mathcal{M}_{0,0}(X, -K_X)$, when X is a del Pezzo surface.

Proposition 2.8 *Let X be a del Pezzo surface of degree two and let K_X be the canonical divisor of X . The scheme $\overline{\mathcal{M}}_{bir}(X, -K_X)$ is isomorphic to a smooth plane quartic.*

Proof. We know (Lemma 2.6) that there is a morphism $\kappa : X \rightarrow \mathbb{P}^2$ associated to the anticanonical sheaf and since $(-K_X)^2 = 2$ (and $-K_X$ is ample), this morphism is finite of degree two. Let $R \subset \mathbb{P}^2$ be the branch curve, and let $2r$ be its degree; denote by $\bar{R} \subset X$ the ramification divisor. Let

$\mathcal{O}_X(1) = \kappa^* \mathcal{O}_{\mathbb{P}^2}(1) \simeq \mathcal{O}_X(-K_X)$; then using the identity $K_X = \kappa^* K_{\mathbb{P}^2} + \bar{R}$, we have $\mathcal{O}_X(-1) \simeq \mathcal{O}_X(-3 + r)$ and we deduce that $r = 2$. Thus R is a plane quartic. It is smooth since the morphism κ has degree two and X is smooth.

The general point of every irreducible component of $\overline{\mathcal{M}}_{bir}(X, -K_X)$ corresponds to a singular divisor in $|-K_X|$. These in turn are parameterized by the tangent lines to the ramification curve R of κ . Let $p \in R$ be a point and let L_p be the tangent line to R at p . It is easy to convince oneself that by associating to each point p in R the morphism which is the normalization of $\kappa^{-1}(L_p)$ at $\kappa^{-1}(p)$, gives an isomorphism $R \simeq \overline{\mathcal{M}}_{bir}(X, -K_X)$. \square

We now deal with the three spaces $\overline{\mathcal{M}}_{bir}(X, -nK)$ for $n \in \{1, 2, 3\}$, where X is a del Pezzo surface of degree one.

Proposition 2.9 *Let X be a del Pezzo surface of degree one and let K_X be the canonical divisor of X . The scheme $\overline{\mathcal{M}}_{0,0}(X, -K_X)$ has dimension zero and length twelve.* \square

The next two results prove that $\overline{\mathcal{M}}_{bir}(X, -2K)$ and $\overline{\mathcal{M}}_{bir}(X, -3K)$ are irreducible assuming that the del Pezzo surface X is general.

Theorem 2.10 *Let X be a general del Pezzo surface of degree one and let C be the closure of the set of points of $|-2K_X|$ corresponding to reduced curves whose normalization is irreducible and of genus zero. Then C is a smooth irreducible curve.*

Proof. The linear system associated to the line bundle $\mathcal{O}_X(-2K_X)$ on the del Pezzo surface of degree one is base-point free and determines a finite morphism κ of degree two to \mathbb{P}^3 , whose image is a quadric cone Q . The image R of the ramification divisor of κ is a smooth canonically embedded curve of genus four which does not contain the vertex of the cone. Clearly, the vertex v of Q is an isolated ramification point, since X is smooth.

Let $C \subset |-2K_X| \simeq \mathbb{P}^3$ be the closure of the set of all points corresponding to integral curves whose normalization is irreducible and of genus zero. In order to prove that C is smooth and irreducible, we will first prove it is connected, and then that it is smooth.

Since the arithmetic genus of a divisor D in $|-2K_X|$ is two, in order for D to be integral and have geometric genus zero, D must be tangent to the ramification divisor \bar{R} at two points. If we translate this in terms of the image of the morphism κ , this implies that the plane P corresponding to the divisor D intersects R along a divisor of the form $2(p) + 2(q) + (r) + (s)$, for

some points $p, q, r, s \in R$. The condition that D should be integral translates to the requirement that the plane P should not contain a line of Q . If this happens, then we have $P \cap R = 2((p) + (q) + (r))$ and $2((p) + (q) + (r))$ is the (scheme-theoretic) fiber of the projection $p|_R$ away from the vertex v (alternatively, $(p) + (q) + (r)$ is the scheme-theoretic intersection of a line on Q with R).

Consider the smooth surface $R \times R$ and the two projection morphisms

$$\begin{array}{ccc} & R \times R & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ R & & R \end{array}$$

where π_1 is the projection onto the first factor and π_2 onto the second. Denote by $\Delta \subset R \times R$ the diagonal. Let $\mathcal{F} := \mathcal{O}_{R \times R}(\pi_2^* K_R - 2\Delta)$ be a sheaf on $R \times R$ and let $\mathcal{E} := (\pi_1)_* \mathcal{F}$ be a sheaf on R .

Clearly \mathcal{F} is invertible. The sheaf \mathcal{E} is locally free of rank two. To prove this, we compute for any $p \in R$

$$h^0(p, \mathcal{F}) := \dim H^0((\pi_1)^{-1}(p), \mathcal{F}|_{(\pi_1)^{-1}(p)}) = \dim H^0(R, \mathcal{O}_R(K_R - 2(p)))$$

We know that the last dimension is at least two, since there is a pencil of planes in \mathbb{P}^3 containing the tangent line to R at p . By Riemann-Roch it follows that the sheaf $\mathcal{O}_R(K_R - 2(p))$ has non-vanishing first cohomology group. By Clifford's Theorem ([Ha] Theorem IV.5.4) the dimension of $H^0(R, \mathcal{O}_R(K_R - 2(p)))$ is at most 3 and since R is not hyperelliptic (because it is a canonical curve) and obviously $K_R - 2(p)$ is not 0 nor K_R , it follows that $h^0(p, \mathcal{F}) = 2$ for all $p \in R$.

We may now apply the first part of Grauert's Theorem ([Ha] Corollary III.12.9) to conclude that $\mathcal{E} = (\pi_1)_* \mathcal{F}$ is locally free and the second part of the same theorem to conclude that the natural morphism of sheaves on $R \times R$

$$\pi_1^* \mathcal{E} = \pi_1^* ((\pi_1)_* \mathcal{F}) \longrightarrow \mathcal{F}$$

is surjective. In turn, this implies ([Ha] Proposition II.7.12) that there is a commutative diagram

$$\begin{array}{ccc} R \times R & \xrightarrow{\varphi} & \mathbb{P}(\mathcal{E}) \\ \pi_1 \downarrow & & \downarrow \pi \\ R & \xrightarrow{id} & R \end{array}$$

The morphism φ is finite of degree four. Let $\bar{C} \subset R \times R$ be the ramification divisor of φ and let $F \subset R \times R$ be the closure of the set of points $\{(p, q) \mid p_v(p) = p_v(q), p \neq q\}$ (remember that p_v is the projection away from the cone vertex v of Q). Note that \bar{C} does not contain any fiber of π , since all the induced morphisms $\varphi_p : R_p := (\pi_1)^{-1}(p) \rightarrow \mathbb{P}_p^1 := \pi^{-1}(p)$ are ramified covers of degree 4. Moreover we have $R_p \cdot \bar{C} = 14$, since for all p such intersection represents the ramification divisor of the morphism φ_p which has degree four, and we may therefore compute the intersection using the Hurwitz formula.

By definition, \bar{C} is the set of pairs (p, q) such that if we denote by P_p^q the plane containing q and the tangent line to R at p (or the osculating plane to R at p , if $p = q$), then we have $P_p^q \cap R \geq 2((p) + (q))$.

We clearly have $F \subset \bar{C}$, since if $(p, q) \in F$ then P_p^q is in fact the tangent plane to Q at p and thus $P_p^q \cap R = 2((p) + (q) + (r)) \geq 2((p) + (q))$.

By definition we have $C \subset \bar{C}$ and no component of C is also a component of F , since the plane corresponding to a point in F intersects Q in a non-reduced curve. It is also immediate to check that in fact C is the residual curve to F in \bar{C} , that is we have $\bar{C} = C \cup F$.

We now prove that the residual curve C to F in \bar{C} is connected and (for general X) smooth.

The connectedness of C is a consequence of a theorem of Kouvidakis: the divisor class of C in $R \times R$ is $4(F_1 + F_2) - \Delta$, where Δ is the diagonal and the F_i 's are the fibers of the two projections to R . Thanks to [La] Example 1.5.13, we know that C is an ample divisor. In particular, C is connected.

To prove the smoothness of C , we will show that for any point $(p, q) \in C$ the two numbers $\text{mult}_{(p,q)}((\pi_1|_C)^{-1}(p))$ and $\text{mult}_{(p,q)}((\pi_2|_C)^{-1}(q))$ cannot both be at least two. Since this would be the case if (p, q) were a singular point, the theorem follows.

Let $p \in R$ and let $(p') + (p'')$ be the divisor obtained by intersecting the curve R with the line on Q through p ; we have

$$\begin{aligned} (\pi_1|_{\bar{C}})^{-1}(p) &= \sum_{q \in R_p} \left(\text{mult}_q(P_p^q \cap R) - 1 \right) (p, q) - 2(p, p) \\ (\pi_1|_F)^{-1}(p) &= (p, p') + (p, p'') \\ (\pi_1|_C)^{-1}(p) &= \sum_{q \in R_p} \left(\text{mult}_q(P_p^q \cap R) - 1 \right) (p, q) - (p, p') - (p, p'') - 2(p, p) \end{aligned}$$

and thus we deduce that

$$\begin{aligned}
 \text{Ram}(\pi_1|_C) &= \text{Ram}(\pi_1|_{\bar{C}}) - \text{Ram}(\pi_1|_F) = \\
 &= \sum_{p \in R} \left(\sum_{q \in R_p \cap \bar{C}} \left(\text{mult}_q(P_p^q \cap R) - 2 \right) (p, q) \right) - \\
 &\quad - \left((p_1, p'_1) + \dots + (p_{12}, p'_{12}) \right)
 \end{aligned}$$

where $R_{p_i} \cap F = 2(p_i, p'_i) + (p_i, p_i)$ (equivalently, the line L_i on Q containing p_i is tangent to the image of the ramification divisor of κ at $p'_i \neq p_i$).

We conclude that $(p, q) \in C$ is a ramification point for $\pi_1|_C$ if and only if $P_p^q \cap R = 2(p) + 3(q) + (r)$, for some $r \in R$. In view of this asymmetry between p and q , we deduce that (p, q) can be a ramification for both projections $\pi_1|_C$ and $\pi_2|_C$ if and only if $P_p^q \cap R = 3(p) + 3(q)$. If p and q are on the same line on Q , then the inverse image under κ of that line would be a cuspidal divisor in $|-K_X|$, which we are excluding. We will now prove that the dimension of the space of smooth canonically embedded curves R of arithmetic genus four lying on a singular quadric and having a plane P transverse to the quadric cone and intersecting R along a divisor of the form $3((p) + (q))$ is at most seven, and thus for the general del Pezzo surface of degree one, this configuration does not happen. This will conclude the proof.

This is simply a dimension count: using automorphisms of \mathbb{P}^3 we may assume that the plane P has equation $X_3 = 0$ and that the quadric cone has equations $X_0X_1 = X_2^2$. We may also assume that p and q have coordinates $[1, 0, 0, 0]$ and $[0, 1, 0, 0]$ respectively. Note that we still have a two-dimensional group of automorphisms (with one generator corresponding to rescaling the coordinate X_3 , and the other corresponding to multiplying the coordinate X_0 by a non-zero scalar and the coordinate X_1 by its inverse). With these choices, the quadric cone is completely determined, as well as the plane P . We still need to compute how many parameters are accounted for by the cubic intersecting the cone in R .

For this last computation, we consider the short exact sequences of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(2) \longrightarrow \mathcal{O}_Q(2) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_Q(2) \longrightarrow \mathcal{O}_Q(3) \longrightarrow \mathcal{O}_R(3) \longrightarrow 0$$

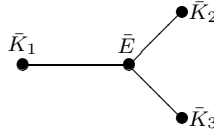
The first sequence implies that the cohomology groups $H^i(Q, \mathcal{O}_Q(2))$ are zero for $i \geq 1$; therefore, from the second sequence we deduce that the

dimension of the space of cubics vanishing on R is nine. Subtracting the two-dimensional automorphism group leaves us with a family of dimension seven. Since there is a family of dimension eight of del Pezzo surfaces of degree one, we conclude. \square

In order to prove a similar result for the divisor class $-3K_X$, we first establish a lemma.

Lemma 2.11 *Let X be a smooth projective surface and let K_1 , K_2 and K_3 be three distinct nodal rational divisors of anticanonical degree one meeting at a point $p \in X$. Suppose that two of the components meet transversely at p . Let $f : \bar{C} := \bar{K}_1 \cup \bar{K}_2 \cup \bar{K}_3 \cup \bar{E} \rightarrow X$ be the stable map of genus zero, such that*

- *the morphism $f_i := f|_{\bar{K}_i}$ is the normalization of K_i followed by the inclusion in X ;*
- *the component \bar{E} is contracted to the point $p \in K_1 \cap K_2 \cap K_3$;*
- *the dual graph of the morphism f is*



Dual graph of f

Then the point represented by the morphism f in $\overline{\mathcal{M}}_{0,0}(X, -3K_X)$ lies in a unique irreducible component.

Proof. The expected dimension of $\overline{\mathcal{M}}_{0,0}(X, -3K_X)$ is $-K_X \cdot (K_1 + K_2 + K_3) - 1 = 2$. The first step of the proof consists of proving that the embedding dimension of $\overline{\mathcal{M}}_{0,0}(X, K_1 + K_2 + K_3)$ at $[f]$ is at most three. To prove this, it suffices to prove that $H^1(\bar{C}, f^*T_X)$ is one-dimensional. This in turn will follow from the fact that $H^0(\bar{C}, f^*T_X)$ has dimension six. On each irreducible component \bar{K}_i we have $f_i^*T_X \simeq \mathcal{O}_{\bar{K}_i}(2) \oplus \mathcal{O}_{\bar{K}_i}(-1)$, where the $\mathcal{O}_{\bar{K}_i}(2)$ summand is the tangent sheaf of \bar{K}_i . Denote by f_E the restriction of f to the component \bar{E} ; we have $f_E^*T_X \simeq \mathcal{O}_{\bar{E}} \oplus \mathcal{O}_{\bar{E}}$. Consider the sequence

$$0 \rightarrow f^*T_X \rightarrow f_1^*T_X \oplus f_2^*T_X \oplus f_3^*T_X \oplus f_E^*T_X \rightarrow T_{X,p} \oplus T_{X,p} \oplus T_{X,p} \rightarrow 0 \quad (2.16)$$

Since two of the K_i 's meet transversely at p , it follows that in order for the global sections on the irreducible components of \bar{C} to glue together, it is

necessary that the sections on the \bar{K}_i 's vanish at the node with \bar{E} . Moreover, if such a condition is satisfied, clearly the sections on the components \bar{K}_i together with the zero section on \bar{E} glue to give a global section of $f^*\mathcal{T}_X$. We deduce that $H^0(\bar{C}, f^*\mathcal{T}_X)$ has dimension six, and it is isomorphic to $H^0(\bar{K}_1, \mathcal{T}_{\bar{K}_1}(-\bar{p}_1)) \oplus H^0(\bar{K}_2, \mathcal{T}_{\bar{K}_2}(-\bar{p}_2)) \oplus H^0(\bar{K}_3, \mathcal{T}_{\bar{K}_3}(-\bar{p}_3))$, where $\bar{p}_i \in \bar{K}_i$ is the node with \bar{E} . From the exact sequence (2.16) and the fact that $H^1(\bar{C}, f_1^*\mathcal{T}_X \oplus f_2^*\mathcal{T}_X \oplus f_3^*\mathcal{T}_X \oplus f_E^*\mathcal{T}_X) = 0$, we deduce that

$$h^1(\bar{C}, f^*\mathcal{T}_X) = 6 - \chi(\bar{C}, f^*\mathcal{T}_X) = 6 - (3 + 3 + 3 + 2) + 6 = 1$$

Thus the embedding dimension of $\overline{\mathcal{M}}_{0,0}(X, K_1 + K_2 + K_3)$ at $[f]$ is at most three, as stated above. It follows that we may write

$$\hat{\mathcal{O}}_{[f]} \overline{\mathcal{M}}_{0,0}(X, K_1 + K_2 + K_3) \simeq k[[t_1, t_2, t_3]]/(g)$$

We thus deduce that all the components through $[f]$ have dimension equal to two, since there is a component of dimension two through $[f]$ and if there were also a component of dimension three or more containing $[f]$, then the embedding dimension would be more than three. Moreover, if there are two components containing $[f]$, then the singular points of $\overline{\mathcal{M}}_{0,0}(X, K_1 + K_2 + K_3)$ near $[f]$ must have dimension equal to one. We prove that $[f]$ is an isolated singular point, and thus we conclude that there is a unique component containing $[f]$.

Let $U \subset \overline{\mathcal{M}}_{0,0}(X, K_1 + K_2 + K_3)$ be the open subset of morphisms $g : \bar{D} \rightarrow X$ which are immersions and birational to their image.

The subset U is contained in the smooth locus of $\overline{\mathcal{M}}_{0,0}(X, K_1 + K_2 + K_3)$ thanks to Proposition 1.17. Moreover $U \cup \{[f]\}$ is a neighbourhood of $[f]$: all the morphisms in a neighbourhood of $[f]$ must have image consisting of at most two components, since the morphisms f_i have no infinitesimal deformations. It follows that there are neighbourhoods of $[f]$ such that $[f]$ is the only morphism with a contracted component. Since the image of f has no cusps and any two components meet transversely, the same statement holds for all the morphisms in a neighbourhood of $[f]$. It follows that $U \cup \{[f]\}$ is a neighbourhood of $[f]$. Thus $[f]$ is an isolated singular point (possibly a smooth point) and since the embedding dimension of $\overline{\mathcal{M}}_{0,0}(X, K_1 + K_2 + K_3)$ at $[f]$ is at most three it follows that $\overline{\mathcal{M}}_{0,0}(X, K_1 + K_2 + K_3)$ is locally irreducible near $[f]$, thus concluding the proof of the lemma. \square

Theorem 2.12 *Let X be a del Pezzo surface of degree one such that the space $\overline{\mathcal{M}}_{\text{bir}}(X, -2K_X)$ is irreducible and all the rational divisors in $|-K_X|$ are nodal. Let S be the closure of the set of points of $|-3K_X|$ corresponding*

to reduced curves whose normalization is irreducible and of genus zero. Then S is an irreducible surface.

Proof. Let $f : \mathbb{P}^1 \rightarrow X$ be a morphism in $\overline{\mathcal{M}}_{bir}(X, -3K_X)$. Thanks to Proposition 1.22 and Lemma 1.20, we may assume that f is an immersion and that its image contains a general point p of X . In particular it follows that $[f]$ represents a smooth point of $\overline{\mathcal{M}}_{bir}(X, -3K_X)$. We choose the point p to be an independent point (Definition 1.24).

Consider the space of morphisms of $\overline{\mathcal{M}}_{bir}(X, -3K_X)$ in the same irreducible component as $[f]$ which contain the point p in their image, denote this space by $\overline{\mathcal{M}}_{bir}(p)$. It follows immediately from the dimension estimates (1.14) that $\dim_{[f]} \overline{\mathcal{M}}_{bir}(p) = 1$ and that $[f]$ is a smooth point of $\overline{\mathcal{M}}_{bir}(p)$. We may therefore find a smooth irreducible projective curve B , a normal surface $\pi : S \rightarrow B$ and a morphism $F : S \rightarrow X$ such that the induced morphism $B \rightarrow \overline{\mathcal{M}}_{bir}(p)$ is surjective onto the component containing $[f]$. From [Ko] Corollary II.3.5.4, it follows immediately that the morphism F is dominant. We want to show that there are fibers of π that are reducible. This is clear, since the morphism $F^* : \text{Pic}(X) \rightarrow \text{Num}(S)$ is injective, and $\text{Pic}(X)$ has rank nine, while if every fiber of π were a smooth rational curve, it would follow that $\pi : S \rightarrow B$ is a ruled surface ([Ha] V.2) and thus that the rank of $\text{Num}(S)$ is two.

This implies that there must be a morphism $f_0 : \bar{C} \rightarrow X$ with reducible domain in the family of stable maps parametrized by B , and since all such morphisms contain the general point p in their image, the same is true of the morphism f_0 . In particular, since the point p does not lie on any rational curve of anticanonical degree 1, it follows that \bar{C} consists of exactly two components \bar{C}_1 and \bar{C}_2 , where each \bar{C}_i is irreducible and we may assume that $f_0(\bar{C}_1)$ has anticanonical degree one and $f_0(\bar{C}_2)$ has anticanonical degree two. Denote by C_i the image of \bar{C}_i . It also follows from the definition of an independent point and Proposition 1.17 that f_0 represents a smooth point of $\overline{\mathcal{M}}_{bir}(X, -3K_X)$.

There are two possibilities for the divisor C_1 : it is either a (-1) -curve (there are 240 such divisors on X), or it is rational curve in the anticanonical divisor class (there are 12 such divisors on X). We will prove that we may assume that C_1 is a rational divisor in the anticanonical linear system.

Suppose that C_1 is a (-1) -curve and let $C'_1 \subset X$ be the (-1) -curve such that $C_1 + C'_1 = -2K_X$. The curve C_2 is thus an integral curve in the linear system $-3K_X - C_1 = -K_X - C'_1$. It follows that C_2 is in the anticanonical linear system on the del Pezzo surface of degree two obtained by contracting C'_1 .

The morphism $\varphi : X \rightarrow \mathbb{P}^2$ associated to the divisor C_2 is the contraction of the (-1) -curve C'_1 followed by the degree two morphism to \mathbb{P}^2 induced by the anticanonical divisor on the resulting surface X' . In the plane \mathbb{P}^2 we therefore have

- the image of the ramification curve R , which is a smooth plane quartic;
- the image of C'_1 , which is a point q ;
- the image of C_1 , which is a plane quartic with a triple point at q and is everywhere tangent to $\varphi(R)$;
- the image of C_2 , which is a tangent line to $\varphi(R)$.

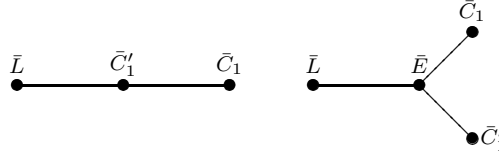
To be precise, the ramification divisor of φ consists of two disjoint components, one is the (-1) -curve C'_1 , whose image is the point q , and the other is a curve whose image is a smooth plane quartic.

Consider the morphism

$$\mathrm{Sl}_{f_0}(\bar{C}_2) \xrightarrow{a} \bar{C}_1$$

and let $\bar{p} \in \bar{C}_1$ be one of the (three) points mapping to the intersection $C_1 \cap C'_1$ (and in particular, $\varphi(f_0(\bar{p})) = q$). Let f_1 be a morphism in the fiber of a above the point \bar{p} . The image of f_1 consists of the divisor $\varphi(C_1)$ together with one of the tangent lines L to $\varphi(R)$ containing the point q .

The domain curve of f_1 consists of possibly a contracted component and three more non-contracted components \bar{C}_1 mapped to C_1 , \bar{L} mapped to the closure of $\varphi^{-1}(L) \setminus C'_1$ and finally \bar{C}'_1 mapped to the (-1) -curve C'_1 . The possible dual graphs of f_1 are



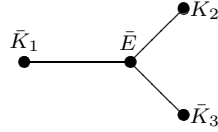
Possible dual graphs of f_1

Note that in the first case f_1 represents a smooth point of the space $\overline{\mathcal{M}}_{0,0}(X, -3K_X)$; in the second case, we may apply Lemma 2.11 to conclude that even if $[f_1]$ is not a smooth point, deforming it produces morphisms in the same irreducible component as f_1 . Smoothing out the components $\bar{C}_1 \cup \bar{C}'_1$ (or $\bar{C}_1 \cup \bar{E} \cup \bar{C}'_1$ if there is a contracted component) we obtain a morphism which has one component (the one obtained by smoothing)

mapped birationally to a rational curve in $|-2K_X|$ and another component (the component \bar{L} , with notation as above) mapped birationally to a rational divisor in $|-K_X|$.

Thus we may deform the original morphism f to a morphism $f_0 : \bar{C}_1 \cup \bar{C}_2 \rightarrow X$ such that \bar{C}_1 is mapped birationally to a rational curve in the anticanonical linear system and \bar{C}_2 is mapped birationally to a rational curve in $|-2K_X|$.

Choose three nodal rational curves K_1 , K_2 and K_3 in the linear system $-K_X$. We prove now that we may deform f_0 without changing the irreducible component of $\overline{\mathcal{M}}_{0,0}(X, -3K_X)$ to the morphism $g : \bar{K}_1 \cup \bar{K}_2 \cup \bar{K}_3 \cup \bar{E} \rightarrow X$ such that \bar{K}_i is the normalization of K_i , \bar{E} is contracted to the point in the intersection $K_1 \cap K_2 \cap K_3$ and the dual graph of g is



Dual graph of g

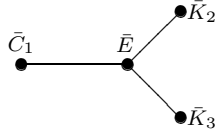
It follows from this and Lemma 2.11 that $\overline{\mathcal{M}}_{bir}(X, -3K_X)$ is irreducible.

To achieve the required deformation, we consider the morphism

$$\mathrm{Sl}_{f_0}(\bar{C}_2) \xrightarrow{\pi} \overline{\mathcal{M}}_{bir}(X, -2K_X)$$

and note that π is surjective since $\overline{\mathcal{M}}_{bir}(X, -2K_X)$ is irreducible by assumption.

Relabeling K_1 , K_2 and K_3 , we may suppose that $C_1 \neq K_2, K_3$. Thus we may specialize f_0 to a morphism $f_1 : \bar{C}_1 \cup \bar{K}_2 \cup \bar{K}_3 \cup \bar{E} \rightarrow X$ such that $f_1(\bar{K}_i) = K_i$, \bar{E} is contracted by f_1 and the dual graph of f_1 is



Dual graph of f_1

Thanks to Lemma 2.11 any deformation of such morphism is in the same irreducible component of $\overline{\mathcal{M}}_{0,0}(X, -3K_X)$ as f_0 and hence in the same irreducible component as the morphism f .

We may now smooth the components $\bar{C}_1 \cup \bar{K}_2 \cup \bar{E}$ to a single irreducible component mapped birationally to the divisor class $-2K_X$ and then we may

use irreducibility of $\overline{\mathcal{M}}_{bir}(X, -2K_X)$ again to prove that we may specialize the component thus obtained to break as $\bar{K}_1 \cup \bar{K}_2$. The morphism g thus obtained is the one we were looking for, and the theorem is proved. \square

Remark. Thanks to Theorem 2.10, the space $\overline{\mathcal{M}}_{bir}(X, -2K_X)$ is irreducible for the general del Pezzo surface of degree one. Thus it follows from Theorem 2.12 that also the space $\overline{\mathcal{M}}_{bir}(X, -3K_X)$ is irreducible for the general del Pezzo surface of degree one.

2.3 The Picard Group and the Orbits of the Weyl Group

In this section we prove some results on the divisor classes of the blow-up of \mathbb{P}^2 at eight or fewer general points. In particular we analyze several questions regarding the divisor classes of the conics and their orbits under the Weyl group.

Let X_δ be the blow-up of \mathbb{P}^2 at $\delta \leq 8$ points such that no three are on a line, no six of them are on a conic and there is no cubic through seven of them with a node at the eighth.

Definition 2.13 *A divisor C on X_δ is called a conic if $-K_{X_\delta} \cdot C = 2$ and $C^2 = 0$.*

Suppose that $\{\ell, e_1, \dots, e_\delta\}$ is a standard basis of $\text{Pic}(X_\delta)$. If $C = a\ell - b_1e_1 - \dots - b_\delta e_\delta$ is a divisor class on X_δ , then to simplify the notation we simply write it as $(a; b_1, \dots, b_\delta)$.

Proposition 2.14 *The conics on X_8 are given, up to permutation of the e_i 's, by the following table:*

Type	ℓ	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
A	1	1	0	0	0	0	0	0	0
B	2	1	1	1	1	0	0	0	0
C	3	2	1	1	1	1	1	0	0
D	4	2	2	2	1	1	1	1	0
E	5	2	2	2	2	2	2	1	0
D'	4	3	1	1	1	1	1	1	1
F	5	3	2	2	2	1	1	1	1
G	6	3	3	2	2	2	2	1	1
H	7	3	3	3	3	2	2	2	1
H'	7	4	3	2	2	2	2	2	2
I	8	4	3	3	3	3	2	2	2
I'	8	3	3	3	3	3	3	3	1
J	9	4	4	3	3	3	3	3	2
K	10	4	4	4	4	3	3	3	3
L	11	4	4	4	4	4	4	4	3

(2.17)

Their numbers are given by the table:

δ	8	7	6	5	4	3	2
<i>conics</i>	2160	126	27	10	5	3	2

Proof. We proceed just like in [Ma] IV, §25. The condition of being a conic translates to the equations

$$\begin{cases} a^2 - \sum_{i=1}^8 b_i^2 = 0 \\ 3a - \sum_{i=1}^8 b_i = 2 \end{cases}$$

and we may equivalently rewrite these as

$$\begin{cases} \sum_{i=1}^8 (a - 2b_i - 2)^2 = 16 \\ 3a - \sum_{i=1}^8 b_i = 2 \end{cases}$$

It is now easy (but somewhat long) to check that (2.17) is the complete list of solutions up to permutations. \square

Remark. The classes of conics on X_δ for $\delta \leq 7$ are obtained from the ones in list (2.17) by erasing $8 - \delta$ zeros and permuting the remaining coordinates. Thus (up to permutations) the first five rows and seven columns describe conics on X_7 , the first three rows and six columns are the conics on X_6 and so on.

We introduce the following notation (which luckily won't be extremely useful, but allows us to name conics!) for the classes of the conics on X_δ ,

$\delta \leq 8$ (we set also $\bar{E} := e_1 + \dots + e_8$):

$$\left\{ \begin{array}{l} A_i = \ell - e_i \\ B_{ijkl} = 2\ell - e_i - e_j - e_k - e_l \\ C_i^{jk} = 3\ell - \bar{E} - e_i + e_j + e_k \\ D_{ijk}^l = 4\ell - \bar{E} - e_i - e_j - e_k + e_l \\ E_i^j = 5\ell - 2\bar{E} + e_i + 2e_j \\ D'_i = 4\ell - \bar{E} - 2e_i \\ F_i^{jkl} = 5\ell - \bar{E} - 2e_i - e_j - e_k - e_l \\ G_{ij}^{kl} = 6\ell - 2\bar{E} - e_i - e_j + e_k + e_l \end{array} \right\} \left\{ \begin{array}{l} H_{ijk}^l = 7\ell - 3\bar{E} + e_i + e_j + e_k + 2e_l \\ (H')_i^l = 7\ell - 2\bar{E} - 2e_i - e_j \\ I_i^{ijk} = 8\ell - 3\bar{E} - e_i + e_j + e_k + e_l \\ I'_i = 8\ell - 3\bar{E} + 2e_i \\ J_{ij}^k = 9\ell - 3\bar{E} - e_i - e_j + e_k \\ K_{ijkl} = 10\ell - 3\bar{E} - e_i - e_j - e_k - e_l \\ L_i = 11\ell - 4\bar{E} + e_i \end{array} \right. \quad (2.18)$$

Denote by \cdot the intersection form on the lattice $\text{Pic}(X_\delta)$. From now on by an automorphism of $\text{Pic}(X_\delta)$ we will always mean a group automorphism of the lattice which preserves the intersection form and the canonical class; we let $W_\delta := \text{Aut}(\text{Pic}(X_\delta), K_{X_\delta}, \cdot)$, and we refer to W_δ as the Weyl group. It will be useful later to know what are the orbits of pairs of conics under the automorphism group W_δ of $\text{Pic}(X_\delta)$.

Lemma 2.15 *The group W_δ , $2 \leq \delta \leq 8$, acts transitively on the conics.*

Proof. We only prove this in the case $\delta = 8$ and it will be clear from the proof that the same argument applies to the other cases.

Choose a standard basis $\{\ell, e_1, \dots, e_8\}$ of $\text{Pic}(X)$; it is enough to prove that the elements in the list (2.17) are in the same orbit, since any permutation of the indices is an element of W_8 .

Introduce the following automorphism of $\text{Pic}(X_8)$:

$$T_{123} : \left\{ \begin{array}{ll} \ell & \mapsto 2\ell - e_1 - e_2 - e_3 \\ e_1 & \mapsto \ell - e_2 - e_3 \\ e_2 & \mapsto \ell - e_1 - e_3 \\ e_3 & \mapsto \ell - e_1 - e_2 \\ e_\alpha & \mapsto e_\alpha \\ 4 \leq \alpha \leq 8 \end{array} \right.$$

and note that applying T_{123} to an element $(a; b_1, \dots, b_8)$ transforms it to

$$(a; b_1, \dots, b_8) \xrightarrow{T_{123}} (2a - b_1 - b_2 - b_3; a - b_2 - b_3, a - b_1 - b_3, a - b_1 - b_2, b_4, \dots, b_8)$$

By inspection, the quantity $2a - b_1 - b_2 - b_3$ for elements in list (2.17) is always strictly smaller than the initial value of a unless $a = 1$. Permuting the indices so that b_1, b_2, b_3 are the three largest coefficients among the b_i 's and iterating this strategy finishes the argument. Note that we are always “climbing up” list (2.17) and the conics on X_7 are the ones above line 5, and are hence preserved by the automorphism T_{123} and the permutations needed. Similar remarks are valid for X_δ , with $3 \leq \delta \leq 6$, and the result is obvious for X_2 , where the automorphism T_{123} is not defined. \square

Remark. It is known ([Ma] Theorem IV.23.9) that the group W_δ is generated by the permutations of the indices of the e_i 's together with the transformation T_{123} .

Suppose now we consider the action of the Weyl group on ordered pairs of conics (Q_1, Q_2) . Clearly the number $Q_1 \cdot Q_2$ is an invariant of this action, and by looking at the list (2.17) it is easy to convince oneself that

$\delta =$	8	7	6	5	4	3	2
$Q_1 \cdot Q_2 \leq$	8	4	2	2	1	1	1

and that all the possible values between 0 and the number given above are attained.

Thus, for example, we know that the action of W_8 on pairs of conics has at least 9 orbits.

If $\delta = 8$, there is one more “invariant” under W_8 of pairs of conics: define a pair (Q_1, Q_2) to be *ample* if $Q_1 + Q_2$ is an ample divisor on X_8 . Since the property of being ample is a numerical property, it follows that it is a property of the W_8 -orbit of the pair.

The next proposition proves that the lower bounds on the number of orbits obtained by considering the intersection product and ampleness (in case $\delta = 8$) of the pair are in fact the correct number of orbits. Indeed, unless $\delta = 8$ it is enough to consider the intersection product, while if $\delta = 8$, there are two orbits with $Q_1 \cdot Q_2 = 4$, and only one of the two consists of ample pairs.

Proposition 2.16 *Let Q_1 and Q_2 be two conics in X_δ , $2 \leq \delta \leq 8$. The intersection product $Q_1 \cdot Q_2$ determines uniquely the orbit of the (ordered) pair (Q_1, Q_2) under W_δ with the only exception of $\delta = 8$ and $Q_1 \cdot Q_2 = 4$ which has exactly two orbits.*

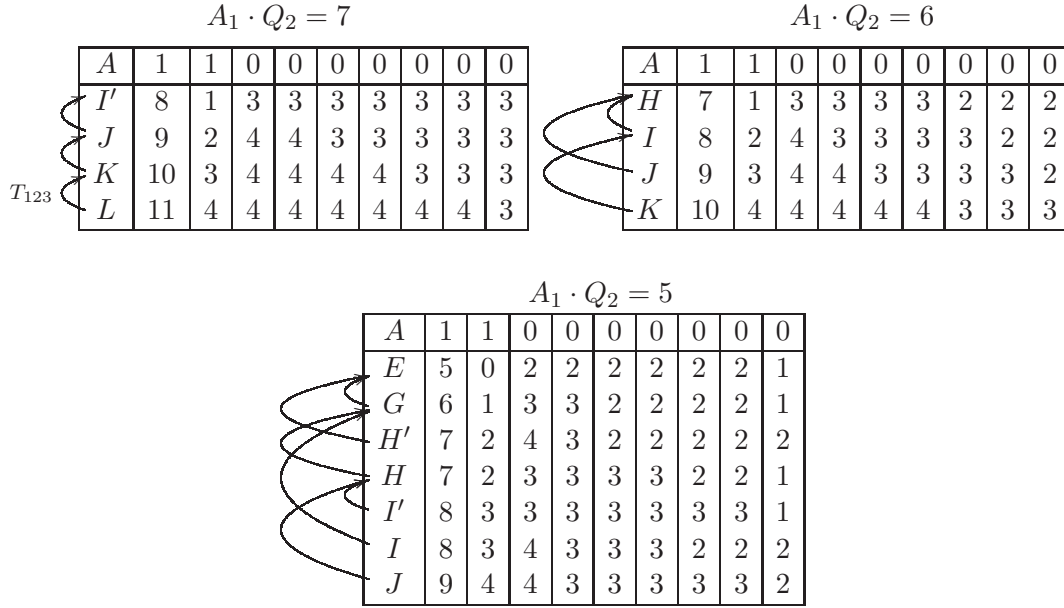
Proof. As for the previous lemma, we will only prove this proposition in the case $\delta = 8$; for the remaining cases simply ignore the nonexistent indices.

Thanks to the previous lemma, we already know that we may assume $Q_1 = \ell - e_1$ which is the conic labeled A_1 in (2.18).

The strategy is very simple: we again climb up the list (2.17) using the automorphism T_{123} followed by a permutation of the indices $\{2, \dots, 8\}$ so that the resulting b_2 and b_3 are the two largest b_i 's, with $i \geq 2$. Note that the elements of W_δ described above do indeed fix Q_1 .

The case $Q_2 = 11\ell - 4\bar{E} + e_1$ is easily seen to be fixed by all permutations of $\{2, \dots, 8\}$ and by the automorphism T_{123} .

In the following diagrams we write all possible conics with the given intersection product with A_1 , sorting the entries b_2, \dots, b_8 in non-increasing order. An arrow going up means: apply T_{123} and permute the indices different from 1 so that the entries under e_2, \dots, e_8 are in non-increasing order.



Next is the case in which there is the exception. Note that if $\delta = 7$, the possible intersection numbers $A_1 \cdot Q_2$ are at most 4, and $A_1 \cdot Q_2 = 4$ only if $Q_2 = 5\ell - e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5 - 2e_6 - 2e_7$; thus the “top orbit” of the next diagram does not appear for $\delta \leq 7$.

$A_1 \cdot Q_2 = 4$									
A	1	1	0	0	0	0	0	0	0
D	4	0	2	2	2	1	1	1	1
F	5	1	3	2	2	2	1	1	1
G	6	2	3	3	2	2	2	1	1
H	7	3	3	3	3	2	2	2	1
I	8	4	3	3	3	3	2	2	2
E	5	1	2	2	2	2	2	2	0
H'	7	3	4	2	2	2	2	2	2

$A_1 \cdot Q_2 = 3$									
A	1	1	0	0	0	0	0	0	0
C	3	0	2	1	1	1	1	1	0
D	4	1	2	2	2	1	1	1	0
E	5	2	2	2	2	2	2	1	0
D'	4	1	3	1	1	1	1	1	1
F	5	2	3	2	2	1	1	1	1
G	6	3	3	2	2	2	2	1	1
H'	7	4	3	2	2	2	2	2	2

$A_1 \cdot Q_2 = 2$									
A	1	1	0	0	0	0	0	0	0
B	2	0	1	1	1	1	0	0	0
C	3	1	2	1	1	1	1	0	0
D	4	2	2	2	1	1	1	1	0
F	5	3	2	2	2	1	1	1	1

$A_1 \cdot Q_2 = 1$									
A	1	1	0	0	0	0	0	0	0
A	1	0	1	0	0	0	0	0	0
B	2	1	1	1	1	0	0	0	0
C	3	2	1	1	1	1	1	0	0
D'	4	3	1	1	1	1	1	1	1

Finally, note that $A_1 + D_{234}^1 = -K_{X_8} + B_{234}$ is ample (being the sum of an ample divisor and a nef divisor), while $(A_1 + E_1^8) \cdot e_8 = 0$. Thus the pair (A_1, D_{234}^1) is ample, while the pair (A_1, E_1^8) is not ample and therefore they cannot lie in the same orbit under the Weyl group. This concludes the proof. \square

Remark. The same statement of Proposition 2.16 is clearly true if we are only interested in unordered pairs of conics. This is obvious because the invariants we needed to detect all the orbits are invariants of unordered pairs, rather than ordered pairs.

The next two lemmas deal with a del Pezzo surface of degree one.

Lemma 2.17 *Let X be a del Pezzo surface of degree one and let $L \subset X$ be a (-1) -curve. If $L_1, L_2 \subset X$ are (-1) -curves such that $L_1 \cdot L = L_2 \cdot L$, then L_1 and L_2 are in the same orbit of the stabilizer of L in $\text{Aut}(\text{Pic}(X))$.*

Remark. The possible intersection numbers between any two (-1) -curves on a del Pezzo surface of degree one are $-1, 0, 1, 2$ and 3 . Moreover, the group $W_8 := \text{Aut}(\text{Pic}(X))$ acts transitively on (-1) -curves ([Ma] Corollary 25.1.1). Thus as a consequence of this fact and the lemma we conclude that the stabilizer in the group W_8 of a (-1) -curve has exactly five orbits on the set of (-1) -curves.

Proof. We may choose a standard basis $\{\ell, e_1, \dots, e_8\}$ of $\text{Pic}(X)$ such that $L = e_8$. Given any divisor class $D \in \text{Pic}(X)$, we write $D = a\ell - b_1e_1 - \dots - b_8e_8$. With these conventions, the classes of the (-1) -curves up to permutations of the indices $1, \dots, 8$ are ([Ma] Table IV.8)

a	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0	-1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	1	1	1	1	1	0	0	0
3	2	1	1	1	1	1	1	0
4	2	2	2	1	1	1	1	1
5	2	2	2	2	2	2	1	1
6	3	2	2	2	2	2	2	2

(2.19)

and the stabilizer of e_8 contains the group S generated by all permutations of $1, \dots, 7$ and the automorphism T_{123} considered above. In fact the stabilizer of e_8 is equal to the group S just described, but we do not need this fact.

The proof consists simply in fixing one value for the coordinate b_8 and checking that all vectors with that last coordinate are in the same orbit of the group S .

$b_8 = 3$. There is only one vector in the list (2.19) with an entry 3 in one of the b_i columns and there is nothing to prove in this case.

$b_8 = 2$. We have

$$\begin{aligned}
T_{123}(6; 3, 2, 2, 2, 2, 2, 2, 2) &= (5; 2, 1, 1, 2, 2, 2, 2, 2) \\
T_{145}(5; 2, 1, 1, 2, 2, 2, 2, 2) &= (4; 1, 1, 1, 1, 1, 2, 2, 2) \\
T_{167}(4; 1, 1, 1, 1, 1, 2, 2, 2) &= (3; 0, 1, 1, 1, 1, 1, 1, 2)
\end{aligned}$$

and using permutations of $1, \dots, 7$ we conclude.

$b_8 = 1$. We have

$$\begin{aligned}
T_{123}(5; 2, 2, 2, 2, 2, 2, 1, 1) &= (4; 1, 1, 1, 2, 2, 2, 1, 1) \\
T_{145}(4; 1, 1, 1, 2, 2, 2, 1, 1) &= (3; 0, 1, 1, 1, 1, 2, 1, 1) \\
T_{456}(3; 0, 1, 1, 1, 1, 2, 1, 1) &= (2; 0, 1, 1, 0, 0, 1, 1, 1) \\
T_{236}(2; 0, 1, 1, 0, 0, 1, 1, 1) &= (1; 0, 0, 0, 0, 0, 0, 1, 1)
\end{aligned}$$

$b_8 = 0$. We have

$$\begin{aligned} T_{123}(3; 2, 1, 1, 1, 1, 1, 0) &= (2; 1, 0, 0, 1, 1, 1, 0) \\ T_{145}(2; 1, 0, 0, 1, 1, 1, 0) &= (1; 0, 0, 0, 0, 0, 1, 0) \\ T_{167}(1; 0, 0, 0, 0, 0, 1, 0) &= (0; -1, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$b_8 = -1$. The only divisor class of a (-1) -curve with $b_8 = -1$ is e_8 .

This completes the cases we needed to check and the proof of the lemma. \square

The following is the last lemma of the section.

Lemma 2.18 *Let L be the divisor class of a (-1) -curve on a del Pezzo surface X of degree one, and let*

$$B := \left\{ \{ \lambda_1, \lambda_2, \lambda_3 \} \mid \lambda_i \text{ is a } (-1)\text{-curve, and } \lambda_1 + \lambda_2 + \lambda_3 = -2K_X + L \right\}$$

The stabilizer in W_8 of L has exactly four orbits on B .

Proof. Choose a standard basis of $\text{Pic}(X)$ such that $L = e_8$. With this choice of basis, we have

$$\lambda_1 + \lambda_2 + \lambda_3 = (6; 2, 2, 2, 2, 2, 2, 1)$$

Let β_i be the coefficient of $-e_8$ in the chosen basis of λ_i . We deduce from above that

$$\beta_1 + \beta_2 + \beta_3 = 1$$

$$-1 \leq \beta_i \leq 3$$

and thus, the solutions $\{\beta_1, \beta_2, \beta_3\}$ of the above system are $\{3, -1, -1\}$, $\{2, -1, 0\}$, $\{1, 1, -1\}$ and $\{1, 0, 0\}$.

Permuting the λ_i 's we may assume that $\beta_1 \geq \beta_2 \geq \beta_3$ and using Lemma 2.17, we may assume that the divisor class of λ_1 is

$$\begin{aligned} (6; 2, 2, 2, 2, 2, 2, 3) &\text{ if } \beta_1 = 3, \\ (6; 3, 2, 2, 2, 2, 2, 2) &\text{ if } \beta_1 = 2, \\ (5; 2, 2, 2, 2, 2, 2, 1) &\text{ if } \beta_1 = 1. \end{aligned}$$

It follows immediately that we must therefore have

$$\begin{aligned}
\beta_1 = 3 : \quad & \begin{cases} \lambda_1 = (6; 2, 2, 2, 2, 2, 2, 2, 3) \\ \lambda_2 = (0; 0, 0, 0, 0, 0, 0, 0, -1) \\ \lambda_3 = (0; 0, 0, 0, 0, 0, 0, 0, -1) \end{cases} \\
\beta_1 = 2 : \quad & \begin{cases} \lambda_1 = (6; 3, 2, 2, 2, 2, 2, 2, 2) \\ \lambda_2 = (0; 0, 0, 0, 0, 0, 0, 0, -1) \\ \lambda_3 = (0; -1, 0, 0, 0, 0, 0, 0, 0) \end{cases} \\
\begin{matrix} \beta_1 = 1 \\ \text{and} \\ \beta_2 = 1 \end{matrix} : \quad & \begin{cases} \lambda_1 = (5; 2, 2, 2, 2, 2, 2, 1, 1) \\ \lambda_2 = (1; 0, 0, 0, 0, 0, 0, 1, 1) \\ \lambda_3 = (0; 0, 0, 0, 0, 0, 0, 0, -1) \end{cases} \\
\begin{matrix} \beta_1 = 1 \\ \text{and} \\ \beta_2 = 0 \end{matrix} : \quad & \begin{cases} \lambda_1 = (5; 2, 2, 2, 2, 2, 2, 1, 1) \\ \lambda_2 = (0; -1, 0, 0, 0, 0, 0, 0, 0) \\ \lambda_3 = (1; 1, 0, 0, 0, 0, 0, 1, 0) \end{cases}
\end{aligned}$$

thus proving the lemma. \square

3 Realizing the Deformation: from Large to Small Degree

3.1 Breaking the Curve

In this section we construct deformations of a general point in every irreducible component of the space $\overline{\mathcal{M}}_{bir}(X, \beta)$ to morphisms with image containing only curves of small anticanonical degree.

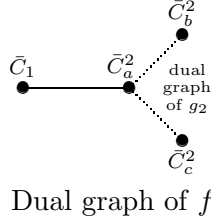
Lemma 3.1 *Let $f : \mathbb{P}^1 \rightarrow X$ be a free birational morphism to a del Pezzo surface. In the same irreducible component of $\overline{\mathcal{M}}_{bir}(X, f_*[\mathbb{P}^1])$ as f there is a morphism $g : \tilde{C} \rightarrow X$ birational to its image such that for every irreducible component $\tilde{C}' \subset \tilde{C}$, $g|_{\tilde{C}'}$ is a free morphism whose image has anticanonical degree two or three.*

Proof. We establish the lemma by induction on $d := -K_X \cdot f_*[\mathbb{P}^1]$. There is nothing to prove if $d \leq 3$, since the image of a free morphism has anticanonical degree at least two (Lemma 1.20).

Suppose that $d \geq 4$. Thanks to Proposition 1.22, we may assume that the image of f contains $d-2 \geq 2$ general points p_1, \dots, p_{d-2} of X . Denote by $\overline{\mathcal{M}}_{bir}(p_1, \dots, p_{d-2})$ the locus of morphisms of $\overline{\mathcal{M}}_{bir}(X, f_*[\mathbb{P}^1])$ whose image contains the points p_1, \dots, p_{d-2} . Using the dimension estimate (1.14), we deduce that $\dim_{[f]} \overline{\mathcal{M}}_{bir}(p_1, \dots, p_{d-2}) = 1$ and thus there is a one-parameter family of morphisms containing f whose images contain the general points p_1, \dots, p_{d-2} . Thanks to Lemmas 1.25 and 1.26 we deduce that in the same irreducible component of $\overline{\mathcal{M}}_{bir}(X, f_*[\mathbb{P}^1])$ as f we can find a morphism $f_0 : \bar{C}_1 \cup \bar{C}_2 \rightarrow X$ such that f_0 is birational to its image, $\bar{C}_i \simeq \mathbb{P}^1$ and $f_0|_{\bar{C}_i}$ is a free morphism. We also have $d_i := -K_X \cdot f_1(\bar{C}_i) \geq 2$, and thus by induction on d , we know that the irreducible component of $\overline{\mathcal{M}}_{bir}(X, f_0(\bar{C}_i))$ containing $f_0|_{\bar{C}_i}$ contains a morphism $g_i : \bar{C}_1^i \cup \dots \cup \bar{C}_{r_i}^i \rightarrow X$ with all the required properties. Thus considering the morphism

$$\mathrm{Sl}_{f_0}(\bar{C}_2) \xrightarrow{\pi} \overline{\mathcal{M}}_{bir}(X, f_0(\bar{C}_2))$$

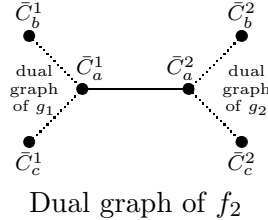
we deduce that we may find a morphism $f_1 : \bar{C}_1 \cup \bar{C}_1^2 \cup \dots \cup \bar{C}_{r_2}^2 \rightarrow X$ with dual graph



Similarly, considering the morphism

$$\mathrm{Sl}_{f_1}(\bar{C}_1) \xrightarrow{\pi} \overline{\mathcal{M}}_{bir}(X, f_0(\bar{C}_1))$$

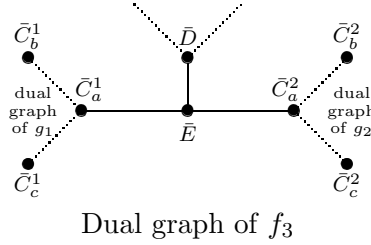
we deduce that we may find a morphism $f_2 : \bar{C}_1^1 \cup \dots \cup \bar{C}_{r_1}^1 \cup \bar{C}_1^2 \cup \dots \cup \bar{C}_{r_2}^2 \rightarrow X$ with dual graph



To conclude, we need to show that the images C_a^1 and C_a^2 of \bar{C}_a^1 and \bar{C}_a^2 respectively can be assumed to be distinct.

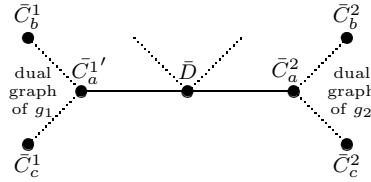
Suppose $C_a^1 = C_a^2$. If the anticanonical degree of C_a^1 is at least three, then we may deform one of them, keeping the image of the node between

\bar{C}_a^1 and C_a^2 fixed and conclude. Suppose therefore that $-K_X \cdot C_a^1 = 2$. Let $\varphi : \bar{C}_a^1 \rightarrow \bar{C}_a^2$ be the morphism $(f_2)^{-1} \circ (f_2)|_{\bar{C}_a^1}$ and let $\bar{p}_i \in \bar{C}_a^i$ be the point in the intersection $\bar{C}_a^1 \cap \bar{C}_a^2$. There are two possibilities: either $\varphi(\bar{p}_1) \neq \bar{p}_2$, or $\varphi(\bar{p}_1) = \bar{p}_2$. In the first case, the deformations of the morphism $f_2|_{\bar{C}_a^1 \cup \bar{C}_a^2}$ fixing the component \bar{C}_a^1 actually change the image of the other component, allowing us to conclude. In the second case, there is a one-dimensional space of deformations of the stable map obtained by “sliding the point \bar{p}_i along \bar{C}_a^i .” Moreover, there must be components in the image of f_2 different from C_a^i , since otherwise the morphism f could not have been birational to its image. Thus we may assume that \bar{C}_a^2 is adjacent to a curve mapped to a curve different from $C_a^1 = C_a^2$, call this curve \bar{D} (remember that g_i is birational to its image). Let $\bar{q} \in \bar{C}_a^2$ be the node between \bar{C}_a^2 and \bar{D} . We may slide the node \bar{p}_i until it reaches the point \bar{q} to obtain a morphism f_3 with dual graph



where the component labeled \bar{E} is contracted to the point $f_2(\bar{q})$. Since the sheaf $f_3^* \mathcal{T}_X$ is globally generated on each component of the domain of f_3 it follows that f_3 is a smooth point of $\overline{\mathcal{M}}_{0,0}(X, f_*[\mathbb{P}^1])$.

Clearly the morphism f_3 is also a limit of morphisms f_4 with dual graphs



Dual graphs of the morphisms f_4

where $\bar{C}_a^{1'}$ is mapped to a general divisor linearly equivalent to C_a^1 and transverse to it. This concludes the proof of the lemma. \square

Lemma 3.2 *Let $f : \bar{C} := \bar{C}_1 \cup \dots \cup \bar{C}_r \rightarrow X$ be a stable map of genus zero and suppose that $f_i := f|_{\bar{C}_i}$ is a free morphism. If $f(\bar{C}_1) \cdot f(\bar{C}_2) > 0$, then in the same irreducible component of $\overline{\mathcal{M}}_{0,0}(X, f_*[\bar{C}])$ containing $[f]$ there is a morphism $g : \bar{D}_1 \cup \dots \cup \bar{D}_r \rightarrow X$ such that \bar{D}_1 and \bar{D}_2 are adjacent, $g|_{\bar{D}_i}$ is a free morphism and $f_*[\bar{C}_i] = g_*[\bar{D}_i]$ for all i 's.*

Proof. Renumbering the components of the domain of f , we may assume that the curve $\bar{C}_{12} := \bar{C}_3 \cup \bar{C}_4 \dots \cup \bar{C}_s$ is the connected component of $\bar{C}_3 \cup \bar{C}_4 \dots \cup \bar{C}_r$ which has a point in common with both \bar{C}_1 and \bar{C}_2 . Moreover, we may also assume that no component of \bar{C}_{12} is mapped to a curve in the same divisor class as \bar{C}_1 or \bar{C}_2 .

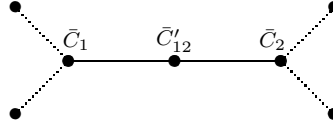
Since all the morphisms $f|_{\bar{C}_i}$ are free, we may deform $f|_{\bar{C}_{12}}$ to a free morphism with irreducible domain \bar{C}'_{12} . Consider the morphism

$$\mathrm{Sl}_f(\bar{C}_{12}) \xrightarrow{\pi} \overline{\mathcal{M}}_{0,0}(X, f_*[\bar{C}_{12}])$$

and note that it is dominant on the component of $\overline{\mathcal{M}}_{0,0}(X, \bar{C}_{12})$ containing $f|_{\bar{C}_{12}}$. Thus we can find a morphism

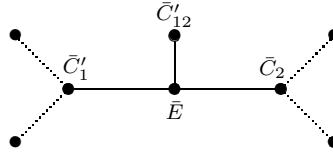
$$f_1 : \bar{C}_1 \cup \bar{C}'_{12} \cup \bar{C}_2 \cup \bar{C}_{s+1} \cup \dots \cup \bar{C}_r \longrightarrow X$$

with dual graph



Dual graph of f_1

We want to deform f_1 to a morphism f_2 with dual graph



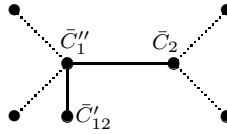
Dual graph of f_2

where \bar{E} is a contracted component. This is immediate considering the morphism

$$\mathrm{Sl}_{f_1}(\bar{C}_1) \xrightarrow{a} \bar{C}'_{12}$$

and noting that it is dominant.

It is clear that we may similarly deform f_2 to a morphism f_3 obtained by sliding \bar{C}'_1 along \bar{C}_2 away from the component \bar{C}'_{12} . The dual graph of the morphism f_3 is



Dual graph of f_3

To conclude we consider the morphism

$$\mathrm{Sl}_{f_3}(\bar{C}'_{12}) \xrightarrow{\pi} \overline{\mathcal{M}}_{0,0}(X, f_*[\bar{C}_{12}])$$

to deform $f_3|_{\bar{C}'_{12}} \simeq f_1|_{\bar{C}'_{12}}$ back to $f|_{\bar{C}_{12}}$ and conclude the proof of the lemma. \square

3.2 Easy Cases: \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ and $Bl_p(\mathbb{P}^2)$

This section proves the irreducibility of the spaces $\overline{\mathcal{M}}_{bir}(X, \alpha)$ where X is a del Pezzo surface of degree eight or nine. Of course, in the case of \mathbb{P}^2 this result is obvious: for a given degree d of the image, the space $\mathrm{Hom}_d(\mathbb{P}^1, \mathbb{P}^2)$ of maps with image of degree d is birational to the set of triples of homogeneous polynomials of degree d up to scaling. Since the space $\mathrm{Hom}_d(\mathbb{P}^1, \mathbb{P}^2)$ dominates $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d[line])$, we deduce the stated irreducibility. Similar considerations apply to $\mathbb{P}^1 \times \mathbb{P}^1$. The result is less obvious for $Bl_p(\mathbb{P}^2)$. We prove the result for $Bl_p(\mathbb{P}^2)$, but similar techniques would also apply to the other two cases.

Note that the same result for the cases \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$ follows also from [KP].

Theorem 3.3 *The spaces of stable maps $\overline{\mathcal{M}}_{bir}(\mathbb{P}^2, \alpha)$, $\overline{\mathcal{M}}_{bir}(\mathbb{P}^1 \times \mathbb{P}^1, \beta)$ and $\overline{\mathcal{M}}_{bir}(Bl_p(\mathbb{P}^2), \gamma)$ are irreducible for all divisor classes α , β and γ .*

Proof. As remarked above, we only treat the case of $Bl_p(\mathbb{P}^2)$. To simplify the notation, let \mathbb{P} denote $Bl_p(\mathbb{P}^2)$. Let $f : \mathbb{P}^1 \rightarrow Bl_p(\mathbb{P}^2)$ be a general morphism in an irreducible component of $\overline{\mathcal{M}}_{bir}(\mathbb{P}, \delta)$.

We first examine the cases $-K_{\mathbb{P}} \cdot f_*[\mathbb{P}^1] \leq 3$ separately.

If $K_{\mathbb{P}} \cdot f_*[\mathbb{P}^1] = -1$, then $f_*[\mathbb{P}^1]$ is the unique (-1) -curve E . In this case we clearly have $\overline{\mathcal{M}}_{0,0}(Bl_p(\mathbb{P}^2), E) = \{[f]\}$.

From now on, we may assume that $f(\mathbb{P}^1)$ is not a (-1) -curve, and thus $f_*[\mathbb{P}^1]$ is a nef divisor, since the only integral curve on \mathbb{P} having negative square is the exceptional divisor.

Suppose that $-K_{\mathbb{P}} \cdot f_*[\mathbb{P}^1] = 2$. It follows easily that $f_*(\mathbb{P}^1)$ is the strict transform of a line in \mathbb{P}^2 . In this case the result is evidently true, since the mapping space is identified with the linear $|f_*(\mathbb{P}^1)|$.

Suppose that $-K_{\mathbb{P}} \cdot f_*[\mathbb{P}^1] = 3$. Let $E \subset Bl_p(\mathbb{P}^2)$ be the exceptional divisor and let $L \subset Bl_p(\mathbb{P}^2)$ be an irreducible divisor representing the class obtained by pulling-back the divisor class of a line in \mathbb{P}^2 . The divisor classes

of L and E generate the Picard group of $Bl_p(\mathbb{P}^2)$. The canonical divisor class on $Bl_p(\mathbb{P}^2)$ is $K = -3[L] + [E]$.

We have $f_*[\mathbb{P}^1] = a \cdot [L] - b \cdot [E]$, with $a \geq b \geq 0$, since $f_*[\mathbb{P}^1]$ is a nef divisor. Moreover we know that $3a - b = 3$, and thus we see that we necessarily have $a = 1$ and $b = 0$. Thus we deduce that $\overline{\mathcal{M}}_{bir}(\mathbb{P}, f_*[\mathbb{P}^1]) \simeq \overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, [L]) \simeq \mathbb{P}^2$, in the case of a divisor of anticanonical degree three.

Suppose that $-K_{\mathbb{P}} \cdot f_*[\mathbb{P}^1] \geq 4$. We may use Lemma 3.1 to deform f to a morphism $f' : \bar{C} \rightarrow \mathbb{P}$ where $\bar{C} = \bar{C}_1 \cup \dots \cup \bar{C}_\ell$ are the irreducible components, all immersed by f' and each having anticanonical degree two or three.

We have a morphism $f' : \bar{C} \rightarrow Bl_p(\mathbb{P}^2)$, birational to its image, such that each component of \bar{C} is mapped to a curve of anticanonical degree two or three. We already saw that this means that each component represents one of the two divisor classes $[L] - [E]$ or $[L]$.

Since $f(\bar{C})$ is connected, if all the components of \bar{C} were mapped to curves whose divisor class is $[L] - [E]$ (there are at least two such components because we are assuming the anticanonical degree of the image is at least four), then they would all have the same image, which is ruled out by the fact that f' is birational to its image. It follows that at least one component of \bar{C} , say \bar{C}_1 , is mapped to the divisor class $[L]$.

Using Lemma 3.2 we may slide all the components of \bar{C} mapped to the divisor class $[L] - [E]$ to be adjacent to the component \bar{C}_1 . After having done this, let $\bar{F}_1, \dots, \bar{F}_l$ denote the components mapped to the divisor class $[L] - [E]$, and let $\bar{C}'_1, \dots, \bar{C}'_k$ denote the components mapped to the divisor class $[L]$, where \bar{C}'_1 is the only component adjacent to all the components \bar{F}_j and no other component \bar{C}'_r is adjacent to any \bar{F}_j .

Consider the subgraph of the dual graph spanned by the components \bar{C}'_r ; this is clearly a tree. Suppose that one of the components adjacent to \bar{C}'_1 is \bar{C}'_2 . Using Lemma 3.2, we may slide all the components adjacent to \bar{C}'_1 (and mapped to $[L]$) to be adjacent to \bar{C}'_2 , making \bar{C}'_1 a leaf of the resulting tree. Similarly, considering the subgraph spanned by the components mapped to the divisor class $[L]$ different from \bar{C}'_1 , we may again assume that \bar{C}'_2 is a leaf, and so on. Eventually we end up with a morphism $g : \bar{D} \rightarrow Bl_p(\mathbb{P}^2)$, where the components of \bar{D} mapped to $[L] - [E]$ are $\bar{F}_1, \dots, \bar{F}_l$ and the components mapped to $[L]$ are $\bar{H}_1, \dots, \bar{H}_k$ and the dual graph of g is

$$\begin{array}{c}
 \bar{F}_1 \bullet \\
 \vdots \\
 \bar{F}_l \bullet
 \end{array}
 \begin{array}{c}
 \diagdown \\
 \bar{H}_1 \bullet \\
 \diagup
 \end{array}
 \begin{array}{c}
 \bar{H}_2 \bullet \\
 \vdots \\
 \bar{H}_{k-1} \bullet \\
 \bar{H}_k \bullet
 \end{array}
 \quad (3.20)$$

Note that there are isomorphisms

$$\begin{aligned}\overline{\mathcal{M}}_{0,0}(Bl_p(\mathbb{P}^2), [L]) &\simeq \overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, [line]) \simeq (\mathbb{P}^2)^\vee \\ &\text{and} \\ \overline{\mathcal{M}}_{0,0}(Bl_p(\mathbb{P}^2), [L] - [E]) &\simeq \mathbb{P}^1\end{aligned}$$

Thus, since $[L] \cdot [L] = 1$ and $[L] \cdot ([L] - [E]) = 1$, we deduce that the space of all morphisms with dual graph (3.20) is birational to $(\mathbb{P}^2)^k \times (\mathbb{P}^1)^l$, and in particular it is irreducible. Since all the components of the morphisms with dual graph (3.20) are free smooth rational curves, it follows that this locus contains smooth points of $\overline{\mathcal{M}}_{bir}(Bl_p(\mathbb{P}^2), f_*[\mathbb{P}^1])$, and therefore we deduce that the space $\overline{\mathcal{M}}_{bir}(Bl_p(\mathbb{P}^2), f_*[\mathbb{P}^1])$ is irreducible. \square

4 Realizing the Deformation: from Small to Large Degree

4.1 Growing from the Conics

In this section we prove some results that allow us to deform unions of conics to divisors which are the anticanonical divisor on a del Pezzo surface dominated by X . These results will be the main building blocks in the proof of Theorem 4.5.

Proposition 4.1 *Let X_δ be a del Pezzo surface of degree $9 - \delta$ such that the spaces $\overline{\mathcal{M}}_{bir}(X_\delta, \beta)$ are irreducible or empty if $-K_{X_\delta} \cdot \beta = 2, 3$. In the case $\delta = 8$, or equivalently if the degree of X_δ is one, suppose also that all the rational divisors in the anticanonical linear system are nodal. Let $f : \bar{Q} \rightarrow X_\delta$ be a morphism from a connected, projective, nodal curve of arithmetic genus zero. Suppose that \bar{Q}_1 and \bar{Q}_2 are the irreducible components of \bar{Q} and that $f_*[\bar{Q}_1]$ and $f_*[\bar{Q}_2]$ are conics. If $f(\bar{Q}_1) \cdot f(\bar{Q}_2) \geq 2$, then in the irreducible component of $\overline{\mathcal{M}}_{0,0}(X_\delta, f_*[\bar{Q}])$ containing $[f]$ there is a morphism $g : \bar{C} \rightarrow X_\delta$ such that*

- all the irreducible components of \bar{C} are immersed and represent nef divisor classes;
- there is a component $\bar{C}_1 \subset \bar{C}$ and a standard basis $\{\ell, e_1, \dots, e_\delta\}$ of $\text{Pic}(X_\delta)$ with

$$g_*[\bar{C}_1] = 3\ell - e_1 - \dots - e_\alpha$$

for some $\alpha \leq \delta$;

- if $g_*[\bar{C}_1] = -K_{X_8}$, then we may choose which of the twelve rational divisors in $| -K_{X_8} |$ the image of \bar{C}_1 is;
- the point $[g]$ is smooth.

Proof. Observe that f represents a smooth point of $\overline{\mathcal{M}}_{0,0}(X_\delta, f_*[\bar{Q}])$, since $f^*\mathcal{T}_X$ is globally generated on both components of \bar{Q} . Note also that by considering $\text{Sl}_f(\bar{Q}_i)$ we may assume that $Q_i := f(\bar{Q}_i)$ misses any preassigned subscheme of X of codimension 2. In particular, we may suppose that Q_i does not contain the intersection points between any two (-1) -curves.

We first take care of the case $Q_1 \cdot Q_2 = 2$: we may assume by Proposition 2.16 that $Q_1 = \ell - e_1$ and $Q_2 = 2\ell - e_2 - e_3 - e_4 - e_5$. It is therefore enough to smooth $\bar{Q}_1 \cup \bar{Q}_2$ to prove the proposition.

This concludes the proof if $\delta \leq 6$ since on a del Pezzo surface of degree at least three there do not exist conics Q_1 and Q_2 such that $Q_1 \cdot Q_2 \geq 3$.

Suppose that $Q_1 \cdot Q_2 \geq 3$. Our first step is to write Q_2 as a sum of two (-1) -curves M_1 and M_2 so that in some standard basis $\{\ell', e'_1, \dots, e'_\delta\}$ we have

$$Q_1 + M_1 = (3\ell' - e'_1 - \dots - e'_\alpha) + N$$

where N is a nef divisor. We assume $Q_1 = A_1 = \ell - e_1$ (Lemma 2.15). Here is the explicit decomposition $Q_2 = M_1 + M_2$ in all the needed cases (Proposition 2.16):

$Q_1 \cdot Q_2 = 3$ $Q_2 = (5; 2, 2, 2, 2, 2, 1, 0)$ $M_1 = (3; 1, 2, 1, 1, 1, 1, 0)$ $M_2 = (2; 1, 0, 1, 1, 1, 1, 0, 0)$	$Q_1 \cdot Q_2 = 4$ $(Q_1 + Q_2) \cdot e_8 = 0$ $Q_2 = (5; 1, 2, 2, 2, 2, 2, 0)$ $M_1 = (3; 1, 2, 1, 1, 1, 1, 0)$ $M_2 = (2; 0, 0, 1, 1, 1, 1, 1, 0)$
$Q_1 \cdot Q_2 = 4$ $Q_1 + Q_2 \text{ ample (on } X_8)$ $Q_2 = (4; 0, 2, 2, 2, 1, 1, 1, 1)$ $M_1 = (3; 0, 2, 1, 1, 1, 1, 1, 1)$ $M_2 = (1; 0, 0, 1, 1, 0, 0, 0, 0)$	$Q_1 \cdot Q_2 = 5$ $Q_2 = (5; 0, 2, 2, 2, 2, 2, 1)$ $M_1 = (3; 0, 2, 1, 1, 1, 1, 1, 1)$ $M_2 = (2; 0, 0, 1, 1, 1, 1, 1, 0)$
$Q_1 \cdot Q_2 = 6$ $Q_2 = (7; 1, 3, 3, 3, 3, 2, 2, 2)$ $M_1 = (3; 0, 2, 1, 1, 1, 1, 1, 1)$ $M_2 = (4; 1, 1, 2, 2, 2, 1, 1, 1)$	$Q_1 \cdot Q_2 = 7$ $Q_2 = (8; 1, 3, 3, 3, 3, 3, 3, 3)$ $M_1 = (3; 0, 2, 1, 1, 1, 1, 1, 1)$ $M_2 = (5; 1, 1, 2, 2, 2, 2, 2, 2)$

$$\begin{array}{c}
Q_1 \cdot Q_2 = 8 \\
\boxed{
\begin{array}{l}
Q_2 = (11; 3, 4, 4, 4, 4, 4, 4) \\
M_1 = (5; 1, 2, 2, 2, 2, 2, 1) \\
M_2 = (6; 2, 2, 2, 2, 2, 2, 3)
\end{array}
}
\end{array}$$

Let us check that the previous decomposition has the required property:

$$\begin{aligned}
Q_1 \cdot Q_2 = 3 \quad Q_1 + M_1 &= T_{127}(3\ell - e_1 - \dots - e_6) \\
Q_1 \cdot Q_2 = 4 \quad Q_1 + M_1 &= \begin{cases} T_{127}(3\ell - e_1 - \dots - e_6) & \text{if } (Q_1 + Q_2) \cdot e_8 = 0 \\ -K_{X_8} + (\ell - e_2) & \text{if } Q_1 + Q_2 \text{ is ample} \end{cases} \\
Q_1 \cdot Q_2 = 5 \quad Q_1 + M_1 &= -K_{X_8} + (\ell - e_2) \\
Q_1 \cdot Q_2 = 6 \quad Q_1 + M_1 &= -K_{X_8} + (\ell - e_2) \\
Q_1 \cdot Q_2 = 7 \quad Q_1 + M_1 &= -K_{X_8} + (\ell - e_2) \\
Q_1 \cdot Q_2 = 8 \quad Q_1 + M_1 &= -K_{X_8} + (3\ell - e_1 - \dots - e_7)
\end{aligned}$$

Next we show that we can deform f so that the dual graph of the resulting morphism f_1 is

$$\begin{array}{ccc}
\bar{Q}_1 & \bar{M}_1 & \bar{M}_2 \\
\bullet & \text{---} & \bullet
\end{array} \tag{4.21}$$

Dual graph of f_1

where of course \bar{M}_i maps to the (-1) -curve with divisor class M_i . To achieve this, consider

$$a : \mathrm{Sl}_f(\bar{Q}_2) \longrightarrow \bar{Q}_1$$

The morphism a is not constant because $f|_{\bar{Q}_2}$ is free, and hence it is surjective. We denote with the symbols M_1 and M_2 both the divisor classes and the (-1) -curves on X with the same divisor class. Let $\bar{p} \in \bar{Q}_1$ be a point such that $f(\bar{p}) =: p \in M_1$; such a point exists, since $Q_1 \cdot M_1 \geq 2$ by inspection.

Thanks to the surjectivity of a , we may find $f_1 : \bar{Q}_1 \cup \bar{Q}'_2 \rightarrow X$ such that $a(f_1) = \bar{p}$, and in particular, the node between \bar{Q}_1 and \bar{Q}'_2 maps to $p \in M_1$. Since $Q_2 \cdot M_1 = 0$ and since $f_1(\bar{Q}'_2) \cap M_1 \ni p$, it follows that $f_1(\bar{Q}'_2) \supset M_1$. Thus we have that $\bar{Q}'_2 = \bar{M}_1 \cup \bar{M}_2$, where \bar{M}_2 maps to the (-1) -curve $M_2 \subset X$; the dual graph of f_1 is the one in (4.21): by construction \bar{Q}_1 and \bar{M}_1 are adjacent, and by connectedness of \bar{Q}_2 it follows that \bar{M}_1 and \bar{M}_2 are adjacent; the assumption that Q_1 does not contain the intersections of two (-1) -curves shows that there cannot be contracted components. Note

that the node between \bar{M}_1 and \bar{M}_2 maps to a node, since the intersection number $M_1 \cdot M_2$ equals one.

Let us check that f_1 represents a smooth point of its moduli space. Thanks to Proposition 1.17, we have that the sheaf $\mathcal{C}_1 := \mathcal{C}_{f_1} \otimes \omega_{\bar{Q}_1 \cup \bar{Q}_2}$, whose global sections represent the obstructions, has degrees given by the following diagram:

$$\begin{array}{ccc} \bar{Q}_1 & \bar{M}_1 & \bar{M}_2 \\ \bullet & \cdots \bullet & \bullet \\ \leq -1 & \leq 0 & -1 \end{array}$$

Multi-degree of \mathcal{C}_1

A solid edge means that the sheaf \mathcal{C}_1 is locally free at the corresponding node, while a dotted edge means that the sheaf \mathcal{C}_1 need not be locally free at that node (we could make sure that the sheaf \mathcal{C}_1 is locally free by reducing to the case in which Q_1 intersects transversely M_1 , but this is not needed). It is now clear that \mathcal{C}_1 has no global sections, and thus the point f_1 is smooth.

We smooth the components $\bar{Q}_1 \cup \bar{M}_1$ to a single irreducible component \bar{Q}'_1 . We obtain a morphism $g' : \bar{Q}'_1 \cup \bar{M}_2 \rightarrow X$, such that in some standard basis $\{\ell', e_1, \dots, e'_\delta\}$ we have $g'_*[\bar{Q}'_1] = (3\ell' - e'_1 - \dots - e'_\alpha) + N$ where $\alpha \geq 6$ and N is a nef divisor. By construction the anticanonical degree of $g'_*[\bar{Q}'_1]$ is three.

In the first two cases above, that is if Q_2 equals $(5; 2, 2, 2, 2, 2, 2, 1, 0)$ or $(5; 1, 2, 2, 2, 2, 2, 2, 0)$, the divisor N above is zero, but in both cases we may write

$$Q'_1 = (3; 1, 1, 1, 1, 1, 1, 1, 0) + (1; 1, 1, 0, 0, 0, 0, 0, 0)$$

We let C_2 be the (-1) -curve with divisor class $(1; 1, 1, 0, 0, 0, 0, 0, 0)$. By inspection we see that $C_2 \cdot M_2 \geq 1$, and therefore we may find a point \bar{c} of \bar{Q}_2 such that $g'(\bar{p}) \in C_2$. Considering the morphism

$$a : \mathrm{Sl}_{g'}(\bar{Q}'_1) \longrightarrow \bar{M}_2$$

we let $g_1 : \bar{C}_1 \cup \bar{C}_2 \cup \bar{M}_2 \longrightarrow X$ be a morphism such that $a(g_1) = \bar{c}$, where we denote by \bar{C}_2 the component mapped to C_2 and by \bar{C}_1 the component mapped to $g'_*[\bar{Q}'_1] - C_2 = (3; 1, 1, 1, 1, 1, 1, 1, 0)$. By construction, the dual graph of g_1 is

$$\begin{array}{ccc} \bar{C}_1 & \bar{C}_2 & \bar{M}_2 \\ \bullet & \cdots \bullet & \bullet \end{array}$$

Dual graph of g_1

Smoothing the components $\bar{C}_2 \cup \bar{M}_2$ we conclude the proof of the proposition in these cases.

In the remaining cases (the ones for which $Q_1 + Q_2$ is ample on a del Pezzo surface of degree one) we write $g'_*[\bar{Q}'_1] = -K_{X_8} + N$, where N is $(\ell - e_2)$, if $Q_1 \cdot Q_2 \leq 7$ and N is $(3\ell - e_1 - \dots - e_7)$, if $Q_1 \cdot Q_2 = 8$.

By assumption the space $\overline{\mathcal{M}}_{bir}(X_\delta, g'_*[\bar{Q}'_1])$ is irreducible. We may therefore deform the morphism g' to a morphism $g_1 : \bar{K} \cup \bar{N} \cup \bar{M}_2 \rightarrow X$, such that \bar{K} is mapped to any preassigned rational divisor in $|-K_{X_8}|$ and \bar{N} is mapped to a general divisor in $|N|$. The possible dual graphs for g_1 are



Possible dual graphs of g_1

We smooth the two components \bar{M}_2 and the one adjacent to it. In either case, the proposition is proved: this is obvious if \bar{N} is adjacent to \bar{M}_2 ; if \bar{K} is adjacent to \bar{M}_2 , note that $-K_{X_8} + M_2$ is the pull-back of the anticanonical divisor on the del Pezzo surface obtained by contracting M_2 . This concludes the proof of the proposition. \square

Remark. The proof above only requires the existence of one nodal rational divisor in $|-K_{X_8}|$.

Proposition 4.2 *Let X_δ be a del Pezzo surface of degree $9 - \delta$. Let $f : \bar{Q} \rightarrow X_\delta$ be a morphism from a connected, projective, nodal curve of arithmetic genus zero. Suppose that $\bar{Q}_1, \bar{Q}_2, \bar{Q}_3$ are the irreducible components of \bar{Q} and that $f_*[\bar{Q}_i]$ is a conic, for all i . If $f(\bar{Q}_i) \cdot f(\bar{Q}_j) = 1$ for all $i \neq j$, then in the irreducible component of $\overline{\mathcal{M}}_{0,0}(X_\delta, f_*[\bar{Q}])$ containing $[f]$ there is an immersion $g : \bar{C} \rightarrow X_\delta$ such that \bar{C} is irreducible and $g_*[\bar{C}] = 3\ell - e_1 - e_2 - e_3$, for some choice of standard basis $\{\ell, e_1, \dots, e_\delta\}$.*

Proof. It is enough to show that we may find a standard basis such that $f_*[\bar{Q}_i] = \ell - e_i$, for $i \in \{1, 2, 3\}$, since then smoothing out all the components we conclude. Denote by Q_i the image of \bar{Q}_i . Thanks to Proposition 2.16 we may assume that $Q_1 = \ell - e_1$ and $Q_2 = \ell - e_2$. Looking at the list (2.17) we easily see that either we may assume that $Q_3 = \ell - e_3$ and we are done, or $Q_3 = 2\ell - e_1 - e_2 - e_3 - e_4$, up to permutations of the coordinates. In this last case, we apply T_{124} to all three divisor classes. Both Q_1 and Q_2 are fixed by T_{124} , while $T_{124}(Q_3) = \ell - e_3$. \square

4.2 Reduction of the Problem to Finitely Many Cases

This section gathers the information obtained in the previous sections to prove that the irreducibility of $\overline{\mathcal{M}}_{bir}(X, \beta)$ for all β can be checked by examining only finitely many cases.

First we prove two simple results.

Lemma 4.3 *Let X be a smooth projective surface and let $D \in \text{Pic}(X)$ be a base-point free nef divisor such that $D^2 > 0$. If N is a nef divisor such that $D \cdot N = 0$, then $N \equiv 0$.*

Proof. Let $\varphi : X \rightarrow \mathbb{P}^n$ be the morphism induced by the linear system $|D|$ and denote by X' the image of φ . We clearly have that $D' := \varphi_*[D]$ is an ample divisor on X' .

The push-forward of a nef divisor N on X is a nef on X' : let $C \subset X'$ be an effective curve; we have $\varphi_*N \cdot C = N \cdot \varphi^*C \geq 0$, since φ^*C is an effective curve.

Let N be a nef divisor on X such that $N \cdot D = 0$. We have $\varphi_*N \cdot D' = N \cdot \varphi^*D' = N \cdot D = 0$, and therefore by the Hodge Index Theorem we deduce that either $\varphi_*N \equiv 0$ or $(\varphi_*N)^2 < 0$. Since φ_*N is nef, it is a limit of ample divisors and it follows that $(\varphi_*N)^2 \geq 0$. We deduce that $\varphi_*N \equiv 0$ and thus that N is numerically equivalent to a linear combination of curves contracted by φ . Since the intersection form on the span of the contracted curves is negative definite and N is nef, we deduce that $N \equiv 0$. \square

Corollary 4.4 *Let X be a del Pezzo surface and let D be a nef divisor on X which is not a multiple of a conic. If a nef divisor N on X is such that $D \cdot N = 0$, then $N = 0$.*

Proof. The result is obvious in the case $X = \mathbb{P}^2$. Thanks to the previous lemma and the fact that numerical equivalence is the same as equality of divisor classes on a del Pezzo surface, it is enough to check that a multiple of a nef divisor class D on X is base-point free and has positive square, unless D is the divisor class of a conic.

Write $D = n_\delta(-K_{X_\delta}) + \dots + n_2(-K_2) + D'$ as in Corollary 2.5. It is immediate to check that $2D$ is base-point free (in fact, unless $D = -K_X$ and X has degree one, then D itself is base-point free). If one of the n_α 's is non-zero, then clearly the square of D is positive (note that all the divisors appearing in the above expression of D are nef and thus effective since X is a del Pezzo surface). If all the n_α 's are zero, then $D = D'$ is a nef divisor on a del Pezzo surface of degree eight.

If $X = \mathbb{P}^1 \times \mathbb{P}^1$, let ℓ_1 and ℓ_2 be the two divisor classes $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$ respectively. Any nef divisor class is a non-negative linear combination of ℓ_1 and ℓ_2 ; thus we may write $D = a_1\ell_1 + a_2\ell_2$, with $a_1, a_2 \geq 0$. Moreover, if one of the a_i 's were zero, then D would be a multiple of a conic: we deduce that $a_i > 0$. Thus we compute $D^2 = 2a_1a_2 > 0$.

If $X = \text{Bl}_p(\mathbb{P}^2)$, let ℓ and e be the pull-back of the divisor class of a line and the exceptional divisor under the blow down morphism to \mathbb{P}^2 respectively. Any nef divisor class is a non-negative linear combination of ℓ and $\ell - e$; thus we may write $D = a\ell + b(\ell - e)$, with $a, b \geq 0$. Moreover, if $a = 0$, then D is a multiple of a conic: we deduce that $a > 0$. Thus we compute $D^2 = a(a + 2b) > 0$ and the proof is complete. \square

We are now ready to prove the main result of the section. The proof involves several steps and is quite long.

Theorem 4.5 *Let X be a del Pezzo surface such that the spaces $\overline{\mathcal{M}}_{\text{bir}}(X, \beta)$ are irreducible (or empty) for all nef divisors β such that $2 \leq -K_X \cdot \beta \leq 3$. In the case $\deg X = 1$, suppose that all the rational divisors in the anticanonical system are nodal. Then, for any nef divisor $D \subset X$ such that $-K_X \cdot D \geq 2$, the space $\overline{\mathcal{M}}_{\text{bir}}(X, D)$ is irreducible or empty.*

Proof. We establish the theorem by induction on $d := -K_X \cdot D$. By hypothesis, the theorem is true if $d \leq 3$.

Suppose that $d \geq 4$. Let $f : \mathbb{P}^1 \rightarrow X$ be a general morphism in an irreducible component of $\overline{\mathcal{M}}_{\text{bir}}(X, D)$. Since the morphism f is a general point on an irreducible component of $\overline{\mathcal{M}}_{\text{bir}}(X, D)$ and $d \geq 2$, it follows that f is an immersion and that it is a free morphism.

If there is a (-1) -curve $L \subset X$ such that $L \cdot D = 0$, then let $b : X \rightarrow X'$ be the contraction of L . We have $\overline{\mathcal{M}}_{\text{bir}}(X, D) \simeq \overline{\mathcal{M}}_{\text{bir}}(X', b_*D)$, and thus we reduce to the case in which the divisor D intersects strictly positively every (-1) -curve. By Theorem 3.3 we may also assume that the degree of X is at most seven. Thus Proposition 2.4 implies that D is an ample divisor.

Thanks to Lemma 3.1 we may deform f to a morphism $g : \bar{C} \rightarrow X$ such that each component $\bar{C}_0 \subset \bar{C}$ is immersed to a curve of anticanonical degree two or three. We want to show that we may specialize g to a morphism in which one component is mapped to a multiple of the divisor class $-K_X$. We will prove this in a series of steps.

Step 1. There is a standard basis $\{\ell, e_1, \dots, e_\delta\}$ of $\text{Pic}(X)$ and a component \bar{C}_1 of \bar{C} mapped birationally either to the divisor class $3\ell - e_1 - \dots - e_\alpha$, for $\alpha \in \{1, \dots, 7\}$, or to $-rK_{X_8}$, for $r \in \{1, 2, 3\}$. If the image of \bar{C}_1 represents $-K_{X_8}$, then we can choose to which of the twelve rational divisors in $|-K_{X_8}|$ the component \bar{C}_1 maps. The morphism is free on all the components of \bar{C} , except on \bar{C}_1 if it represents $-K_{X_8}$.

The divisors of anticanonical degree two on X are

- the divisor $-2K_X$, if $\deg X = 1$;

- the divisor $-K_X$, if $\deg X = 2$;
- the divisor class of a conic.

The divisors of anticanonical degree three on X are

- the divisor $-3K_X$, if $\deg X = 1$;
- the divisor $-K_X - K_{X'}$, if $\deg X = 1$ and X' is obtained from X by contracting a (-1) -curve;
- the divisor $-K_X + C$, if $\deg X = 1$ and C is the class of a conic;
- the divisor $-K_X$, if $\deg X = 3$;
- the divisor ℓ , for some standard basis $\{\ell, e_1, \dots, e_\delta\}$.

Thanks to the irreducibility assumption on the spaces $\overline{\mathcal{M}}_{bir}(X, \beta)$, for $2 \leq -K_X \cdot \beta \leq 3$, we reduce to the case in which all components of \bar{C} are mapped to either the divisor class of a conic or the divisor class ℓ , for some choice of standard basis.

We reduce further to the following case:

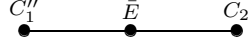
- (\star) There is a standard basis $\{\ell, e_1, \dots, e_\delta\}$ of $\text{Pic}(X)$ such that all curves of degree three in the image of g have divisor class ℓ .

This is easily accomplished. Suppose that \bar{C}_1 and \bar{C}_2 are components of \bar{C} such that $g_*[\bar{C}_1] = \ell_1$ and $g_*[\bar{C}_2] = \ell_2$, where $\{\ell_i, e_1^i, \dots, e_\delta^i\}$ are two standard basis of $\text{Pic}(X)$ and $\ell_1 \neq \ell_2$. We may first of all apply Lemma 3.2 to reduce to the case in which \bar{C}_1 and \bar{C}_2 are adjacent in the dual graph of g . If ℓ_2 were orthogonal to e_1^1, \dots, e_δ^1 , then ℓ_2 would be proportional and hence equal to ℓ_1 . It follows that ℓ_2 is not orthogonal to all the e_j^1 's. By permuting the indices if necessary, we may assume that $\ell_2 \cdot e_1^1 > 0$. Since $g|_{\bar{C}_2}$ is free, we may assume that $g(\bar{C}_2)$ and $E := E_1^1$, the (-1) -curve whose divisor class is e_1^1 , meet transversely. Denote by $\bar{p} \in \bar{C}_2$ a point such that $g(\bar{p}) \in E$. Consider the morphism

$$\text{Sl}_g(\bar{C}_1) \xrightarrow{a} \bar{C}_2$$

and note that it is dominant, since $g|_{\bar{C}_1}$ is free. It follows that we may find a morphism $g_1 : \bar{C}'_1 \cup \bar{C}_2 \cup \dots \cup \bar{C}_r \rightarrow X$ such that $a(g_1) = \bar{p}$. We deduce that $g_1(\bar{C}'_1) \ni p$ and $(g_1)_*[\bar{C}'_1] = \ell_1$. Since $\ell_1 \cdot e_1^1 = 0$, we conclude that

$g_1(\bar{C}'_1)$ contains E and another (irreducible) component whose divisor class is $\ell_1 - e_1^1$. Finally, the subgraph of the dual graph of g_1 spanned by \bar{C}'_1 and \bar{C}_2 is



Subgraph of the dual graph of g_1

where $(g_1)_*[\bar{C}_1''] = \ell_1 - e_1^1$ and $(g_1)_*[\bar{E}] = e_1^1$. We may now smooth $\bar{E} \cup \bar{C}_2$ to a single irreducible component \bar{C}_2' mapped to a curve of anticanonical degree four. With the usual argument of fixing two general points on the image of \bar{C}_2' we may deform the morphism so that \bar{C}_2' breaks in two components each mapped to a curve of anticanonical degree two. The result of this deformation was to replace two components mapped to the different divisor classes ℓ_1 and ℓ_2 by three components mapped to divisor classes of curves of anticanonical degree two.

Iterating the previous argument we deduce that we may assume that condition (\star) holds.

We first treat the case in which there are no components mapped to ℓ . If all the irreducible components of the domain of g are mapped to conics, and two of them have images with intersection number at least two, then we may use Proposition 4.1 to conclude. If all the conics in the image of g have intersection number at most one, then there must be at least three having pairwise intersection products exactly one. Otherwise we would be able to find a standard basis $\{\ell, e_1, \dots, e_\delta\}$ such that the divisor classes of the components of the image of g are in the span of $\ell - e_1$ and $\ell - e_2$. Clearly, no linear combination of these divisors is ample, since

$$(\ell - e_1 - e_2) \cdot (a_1(\ell - e_1) + a_2(\ell - e_2)) = 0$$

Thus there must be at least three components mapped to divisor classes of conics with pairwise intersection products exactly one. Lemma 3.2 allows us to assume that three of these components are adjacent and using Proposition 4.2 we conclude.

Suppose now that there is a component mapped to a curve with divisor class ℓ . Since on a del Pezzo surface of degree at most seven no multiple of ℓ is ample, it follows that there must be components of the domain of g mapped to divisor classes of conics.

Suppose that \bar{C}_1 is mapped to the divisor class ℓ and \bar{C}_2 is mapped to the divisor class of a conic Q . Thanks to Lemma 3.2 we may assume that \bar{C}_1 and \bar{C}_2 are adjacent. Permuting the indices $1, \dots, \delta$ if necessary, we may

assume that the component \bar{C}_2 mapped to the conic $Q = a\ell - b_1e_1 - \dots - b_\delta e_\delta$ has largest possible b_1 . Looking at the table (2.17), it is easy to check that

- if Q is of type H', I, J, K, L , then $Q + e_1 = -K_{X_8} - K_{X'}$, where X' is obtained by contracting a (-1) -curve on X ;
- if Q is of type D', F, G, H, I' , then $Q + e_1 = -K_{X_8} + Q'$, where Q' is a conic;
- if Q is of type C, D, E , then $Q + e_1$ is already of the required form (for a different choice of standard basis, when $Q = D$ or E);
- if Q is of type B , then $Q + \ell$ is of the required form.

If Q is of type B , then we smooth $\bar{C}_1 \cup \bar{C}_2$ to a single irreducible component to conclude. Otherwise, we deform the morphism so that \bar{C}_1 breaks into a component mapped to $\ell - e_1$ and a component e_1 adjacent to \bar{C}_2 . To achieve this splitting, consider the morphism

$$\mathrm{Sl}_g(\bar{C}_1) \xrightarrow{a} \bar{C}_2$$

and let $\bar{p} \in \bar{C}_2$ be a point mapped to the (-1) -curve whose divisor class is e_1 (note that $Q \cdot e_1 \geq 1$). Since the restriction of g to each irreducible component of its domain is free, we may assume that $g(\bar{p})$, as well as the image of the node between \bar{C}_1 and \bar{C}_2 are general points of X . Since the morphism a is dominant, we may find a deformation g' of g such that $a(g') = \bar{p}$. This means that the “limiting component” of \bar{C}_1 breaks in the desired way. Smoothing the union of \bar{C}_2 and the component mapped to e_1 , we obtain a morphism $\bar{g}' : \bar{C}'_1 \cup \bar{C}'_2 \rightarrow X$ where $\bar{g}'_*[\bar{C}'_1] = \ell - e_1$ and $\bar{g}'_*[\bar{C}'_2] = Q + e_1$. The hypotheses of the theorem imply that $\overline{\mathcal{M}}_{bir}(X, \bar{g}'_*[\bar{C}'_2])$ is irreducible since $-K_X \cdot (Q + e_1) = 3$. Thanks to the previous analysis of the divisor class $Q + e_1$, we conclude considering the dominant morphism

$$\mathrm{Sl}_{\bar{g}'}(\bar{C}'_2) \xrightarrow{\pi} \overline{\mathcal{M}}_{bir}(X, \bar{g}'_*[\bar{C}'_2])$$

that we may assume that there is a component of g mapped to the divisor class $3\ell - e_1 - \dots - e_\alpha$, for some $\alpha \leq 8$.

The only remaining case is the one in which the conic Q is of type A . We may therefore suppose that if a component of the domain of g is mapped to the divisor class of a conic, then the divisor class of the image is $\ell - e_j$ for some j . Since the image of g is an ample divisor, it follows that there must be components $\bar{Q}_1, \dots, \bar{Q}_\delta$ mapped to $\ell - e_1, \dots, \ell - e_\delta$ respectively.

Repeated application of Lemma 3.2 allows us to assume that the component mapped to ℓ and the two components \bar{Q}_1, \bar{Q}_2 are adjacent. Smoothing the union of these three components to a single irreducible free morphism, concludes the first step of the proof.

Step 2. There is a component \bar{C}_1 of \bar{C} mapped to the divisor class $-K_{X_\delta}$, if $\delta \leq 7$. If $\delta = 8$ (that is, the degree of $X = X_8$ is one), then \bar{C}_1 mapped to $-rK_{X_8}$, for $r \in \{1, 2, 3\}$. If $r = 1$, then we may choose to which rational divisor in $|-K_{X_8}|$ the component \bar{C}_1 maps.

If the component \bar{C}_1 of **Step 1** is mapped to $-rK_{X_8}$, there is nothing to prove.

Let \bar{C}_1 be the component of g mapped to the divisor class $3\ell - e_1 - \dots - e_\alpha$. If $\alpha = \delta$, then again there is nothing to prove. Suppose therefore that $\alpha < \delta$. There is a component of \bar{C} , say \bar{C}_2 , such that $g_*[\bar{C}_2] \cdot e_{\alpha+1} \geq 1$, since the image of g is an ample divisor; let $C_2 := g_*[\bar{C}_2]$. Moreover, $C_1 := g_*[\bar{C}_1] = 3\ell - e_1 - \dots - e_\alpha$ intersects positively every non-zero nef divisor, thanks to Corollary 4.4. Thus $C_1 \cdot C_2 > 0$ and thanks to Lemma 3.2 we may assume that \bar{C}_1 and \bar{C}_2 are adjacent in the dual graph of g . Since the morphism $g|_{\bar{C}_2}$ is free, we assume also that C_2 meets transversely the (-1) -curve $E_{\alpha+1}$ whose divisor class is $e_{\alpha+1}$. Let $\bar{p} \in \bar{C}_2$ be a point such that $p := g(\bar{p}) \in E_{\alpha+1}$. Consider the morphism

$$\mathrm{Sl}_g(\bar{C}_1) \xrightarrow{a} \bar{C}_2$$

and note that it is dominant, since $g|_{\bar{C}_1}$ is free. It follows that we may find a morphism $g_1 : \bar{C}'_1 \cup \bar{C}_2 \cup \dots \cup \bar{C}_r \rightarrow X$ such that $a(g_1) = \bar{p}$. We deduce that $g_1(\bar{C}'_1) \ni p$ and $(g_1)_*[\bar{C}'_1] = 3\ell - e_1 - \dots - e_\alpha$. Since $(3\ell - e_1 - \dots - e_\alpha) \cdot e_{\alpha+1} = 0$, we conclude that $g_1(\bar{C}'_1)$ contains $E_{\alpha+1}$ and another (irreducible) component whose divisor class is $3\ell - e_1 - \dots - e_{\alpha+1}$. Thus the subgraph of the dual graph of g_1 spanned by \bar{C}'_1 and \bar{C}_2 is

$$\begin{array}{c} \bar{C}'_1 \quad \quad \bar{E}_{\alpha+1} \quad \quad \bar{C}_2 \\ \bullet \text{---} \bullet \text{---} \bullet \end{array}$$

Subgraph of the dual graph of g_1

where $(g_1)_*[\bar{C}'_1] = 3\ell - e_1 - \dots - e_{\alpha+1}$ and $(g_1)_*[\bar{E}_{\alpha+1}] = e_{\alpha+1}$. We may now smooth $\bar{E}_{\alpha+1} \cup \bar{C}_2$ to a single irreducible component \bar{C}'_2 mapped to a curve of anticanonical degree three or four. If this new component has degree four, we break it into two components of anticanonical degree two.

If the degree of X is at least two, iterating this procedure allows us to produce a component of \bar{C} whose image represents the divisor class $-K_X$.

If the degree of X is one, we may apply the same procedure to obtain a component \bar{C}_1 mapped to $-K_{X_8}$, but we still have to prove that we may choose which nodal rational divisor in $|-K_{X_8}|$ is in the image of the morphism.

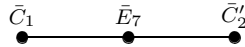
If the component \bar{C}_2 adjacent to the component \bar{C}_1 has degree two, we smooth these two components to a single irreducible one and using the irreducibility of $\overline{\mathcal{M}}_{bir}(X, \beta)$, for $-K_{X_8} \cdot \beta = 3$ we conclude.

If the component \bar{C}_2 adjacent to the component \bar{C}_1 mapped to $-K_{X_8}$ has degree three, then it represents one of the five divisor classes $-3K_{X_8}$, $(-K_{X_8} - K_{X_7})$, $(-K_{X_8} + Q)$, $-K_{X_6}$ or ℓ , where Q is a conic, $-K_{X_7}$ and $-K_{X_6}$ are del Pezzo surfaces of degree two and three respectively dominated by X_8 .

In the first three cases, we deform the morphism so that the component \bar{C}_2 breaks into a component mapped to a preassigned nodal divisor in $|-K_{X_8}|$ and into a component where the morphism is free. In these cases, smoothing the component \bar{C}_1 with the component adjacent to it into which \bar{C}_2 broke finishes the proof.

If \bar{C}_2 is mapped to $-K_{X_6}$ or ℓ , then we may choose a standard basis so that $-K_{X_6} = 3\ell - e_1 - \dots - e_6$. The morphism $\varphi : X_8 \rightarrow \mathbb{P}^2$ determined by the linear system $|\ell|$ is the contraction of the (-1) -curves with divisor classes e_1, \dots, e_8 to the points $q_1, \dots, q_8 \in \mathbb{P}^2$. The image of \bar{C}_1 in \mathbb{P}^2 is a nodal plane cubic through the eight points q_1, \dots, q_8 . The image of \bar{C}_2 is either a rational cubic through q_1, \dots, q_6 or a line. We treat only the case in which the image of \bar{C}_2 is a nodal cubic, since the other one is simpler and the arguments are similar.

Deform the nodal cubic through q_1, \dots, q_6 until it contains a general point $q \in \mathbb{P}^2$. We may now slide the node between \bar{C}_1 and \bar{C}_2 along \bar{C}_1 until it reaches a point on \bar{C}_1 mapped to the point $q_7 \in \mathbb{P}^2$. As we slide the node, we let the image of \bar{C}_2 always contain the general point q . When the deformation is finished, the component \bar{C}_2 breaks as the (-1) -curve with divisor class e_7 and the divisor class $-K_{X_6} - e_7$. Since the point q is general, we know that there are only finitely many (in fact twelve) possible configurations for these limiting positions and we may assume that they are all transverse to the image of \bar{C}_1 . Thus the dual graph of the resulting morphism $g' : \bar{C}_1 \cup \bar{C}'_2 \cup \bar{E}_7 \rightarrow X_8$ is



Dual graph of g'

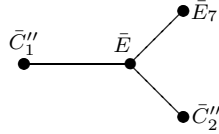
We may smooth the components $\bar{C}_1 \cup \bar{E}_7$ to a unique irreducible component mapped to a curve of anticanonical degree two. The assumptions of the

theorem imply that $\overline{\mathcal{M}}_{bir}(X_8, C_1 + e_7)$ is irreducible and we may therefore deform the morphism so that its domain breaks as a preassigned rational nodal divisor C'' in $|-K_{X_8}|$ and a curve mapped to the divisor class e_7 . The dual graph of the resulting morphism g'' is of one of the following types:



Possible dual graphs of g''

In the first case we smooth $\bar{E}_7 \cup \bar{C}_2''$ to a single irreducible component and conclude. In the second case, we slide the node between \bar{C}_1'' and \bar{C}_2'' until it reaches the node between \bar{C}_1'' and \bar{E}_7 , in such a way that the limiting position of \bar{C}_2'' does not coincide with the image of \bar{C}_1'' (we can do this thanks to the irreducibility of $\overline{\mathcal{M}}_{bir}(X_8, C_1'' + C_2'')$). It follows that the dual graph of the morphism \bar{g} thus obtained is



Dual graph of \bar{g}

It is easy to check that \bar{g} represents a smooth point of $\overline{\mathcal{M}}_{0,0}(X_8, C_1'' + C_2'' + e_7)$, since the sheaf $\bar{g}^* \mathcal{T}_{X_8}$ is globally generated on \bar{C}_2'' and has no first cohomology group on the remaining components, thanks to Lemmas 1.4 and 1.7. We may now smooth the components $\bar{E}_7 \cup \bar{E} \cup \bar{C}_2''$ to a single irreducible component on which the morphism is free to conclude.

A similar and simpler argument proves the same result if \bar{C}_2 has divisor class ℓ . This finishes the proof of this step.

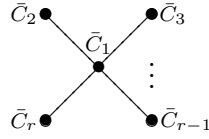
Step 3. We may deform $g : \bar{C} \rightarrow X$ to a morphism $h : \bar{D}_1 \cup \bar{D}_2 \rightarrow X$ where \bar{D}_1 and \bar{D}_2 are irreducible, $h_*[\bar{D}_1] = -rK_X$ ($r \in \{1, 2, 3\}$), $h(\bar{D}_1) \neq h(\bar{D}_2)$ and $h|_{\bar{D}_2}$ is free. If the degree of X is at least two, then $r = 1$. If the degree of X is one and $r = 1$, we may choose which rational divisor in $|-K_X|$ $h(\bar{D}_1)$ is. Note that we are not requiring $h|_{\bar{D}_2}$ to be birational to its image.

Thanks to the previous steps, we may assume that $g_*[\bar{C}_1] = -rK_X$ (with the required restriction for r) and that all the components of \bar{C} different from \bar{C}_1 are immersed to curves of anticanonical degree two or three. Let C_2, \dots, C_r be the components of the image of g different from $g(\bar{C}_1)$, and let \bar{C}_i be the component of \bar{C} whose image is C_i .

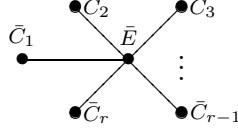
The divisor class $C_2 + \dots + C_r$ is nef and if it is not a multiple of a conic, then it meets all nef curves positively, thanks to Corollary 4.4. Thus, still

assuming that $C_2 + \dots + C_r$ is not a multiple of a conic, we may deform the morphism using Lemma 3.2 and assume that the union of all the components of the domain of g different from \bar{C}_1 is connected. Smoothing the resulting union $\bar{C}_2 \cup \dots \cup \bar{C}_r$ concludes the proof of this step in this case.

Suppose that $C_2 + \dots + C_r$ is a multiple of a conic. Then it follows that $C_2 = \dots = C_r = C$ is a conic. Since g is birational to its image and two divisors linearly equivalent to the same conic are either disjoint or they coincide, it follows that the dual graph of g must be

Dual graph of g

By sliding each \bar{C}_i along \bar{C}_1 , we may assume that the images of all the components mapped to a conic coincide and that the nodes in the source curve all map to the same general point $p \in X$. Thus the dual graph of the resulting morphism g'' is

Dual graph of g''

where \bar{E} is a contracted component whose image is p . The morphism g'' is a smooth point of $\overline{\mathcal{M}}_{bir}(X, g_*[\bar{C}])$ since the sheaf $(g'')^*T_X$ is globally generated on each irreducible component of the domain curve of g'' .

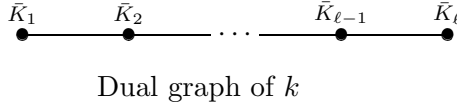
We may smooth all the components $\bar{E} \cup \bar{C}_2 \cup \dots \cup \bar{C}_r$ to a single irreducible component which represents a multiple cover (in fact a degree $r-1$ cover) of its image. The resulting morphism $h : \bar{D}_1 \cup \bar{D}_2 \rightarrow X$ is therefore such that the image of \bar{D}_1 is a rational divisor in the linear system $|-rK_X|$ ($r=1$ unless the degree of X is one, in which case $r \leq 3$), which is an arbitrary preassigned one in case $\deg X = 1$ and $r=1$, and the morphism $h|_{\bar{D}_2}$ is a multiple cover of the divisor class of a conic. This concludes the proof of the third step.

We may write $h_*[\bar{D}_2] = n_8(-K_{X_8}) + \dots + n_2(-K_2) + D'_2$ as in Corollary 2.5 (to simplify the notation we will assume that $\deg X = 1$; if this is not the case, simply set to zero all the coefficients n_α , with $\alpha > 9 - \deg X$). Let

$n := \lfloor \frac{n_8}{2} \rfloor$, if $n_8 \neq 1$ and let $n = 1$, if $n_8 = 1$. Thus we have $n_8 = 2(n-1) + 3$, if n_8 is odd and at least three, and $n_8 = 2n$, if n is even.

Step 4. Let $S \subset X$ be a nodal rational divisor in the linear system $|-K_X|$. We may deform h to a morphism $k : \bar{K} := \bar{K}_1 \cup \dots \cup \bar{K}_\ell \longrightarrow X$ with the following properties:

- P1) the morphism k restricted to each irreducible component of \bar{K} is free, except possibly on \bar{K}_1 ;
- P2) each irreducible component of \bar{K} represents one of the divisor classes $-3K_{X_8}, -2K_{X_8}, -K_{X_7}, \dots, -K_{X_2}, D'_2$, except \bar{K}_1 , whose image may also be S ;
- P3) the dual graph of k is



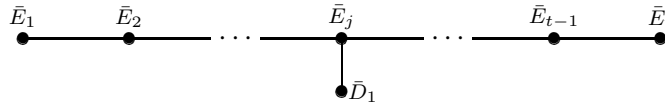
- P4) the component \bar{K}_1 is mapped to $-3K_{X_8}$ if n_8 is odd and at least three, to S if $n_8 = 1$ and to $-2K_{X_8}$ if n_8 is even and bigger than zero;
- P5) the components $\bar{K}_2, \bar{K}_3, \dots, \bar{K}_n$ are mapped to $-2K_{X_8}$;
- P6) let $N_\alpha := n + n_7 + \dots + n_{\alpha+1}$; the components $\bar{K}_{N_\alpha+1}, \dots, \bar{K}_{N_{\alpha-1}}$ are mapped to $-K_{X_\alpha}$;
- P7) if $D'_2 \neq 0$, then \bar{K}_ℓ is mapped to D'_2 ;
- P8) the morphism $k|_{\bar{K}_1 \cup \dots \cup \bar{K}_{\ell-1}}$ is birational to its image.

We call a morphism satisfying all the above properties a *morphism in standard form*.

By induction on the anticanonical degree of the divisor, we know that the space $\overline{\mathcal{M}}_{bir}(X, h_*[\bar{D}_2])$ is irreducible (or empty if $h_*[\bar{D}_2]$ is a multiple of a conic and it is clear how to proceed in this case; we will not mention this issue anymore). We may therefore deform the morphism $h|_{\bar{D}_2}$ to a morphism $l : \bar{E} := \bar{E}_1 \cup \dots \cup \bar{E}_t \longrightarrow X$ in standard form. Considering the morphism

$$\mathrm{Sl}_h(\bar{D}_2) \xrightarrow{\pi} \overline{\mathcal{M}}_{bir}(X, h_*[\bar{D}_2])$$

we may find a deformation \tilde{l} of h such that $\pi(\tilde{l}) = l$. The dual graph of \tilde{l} is



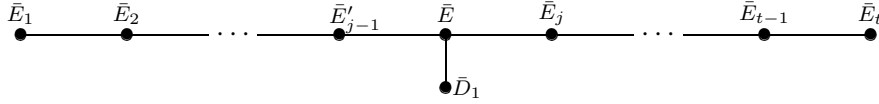
Dual graph of \tilde{l}

for some $1 \leq j \leq t$. We want to show by induction on j that we may assume that $j = 1$. In the case $j = 1$ there is nothing to prove.

Suppose $j \geq 2$ and consider the morphism

$$\mathrm{Sl}_t(\bar{E}_{j-1}) \xrightarrow{a} \bar{E}_j$$

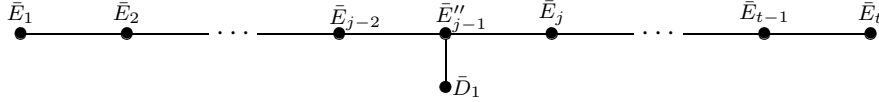
Unless $j = 2$ and \bar{E}_1 represents $|-K_{X_8}|$, the morphism a is dominant and we may find a morphism l_1 such that $a(l_1)$ is the node between \bar{D}_1 and \bar{E}_j . The dual graph of the morphism l_1 is



Dual graph of l_1

and the component \bar{E} is contracted. The morphism l_1 represents a smooth point of $\overline{\mathcal{M}}_{bir}(X, h_*[\bar{C}])$, since $l_1^* \mathcal{T}_X$ is globally generated on all components different from \bar{D}_1 , and $H^1(\bar{D}_1, l_1^* \mathcal{T}_X) = 0$.

The morphism l_1 is also a limit of morphisms with dual graph



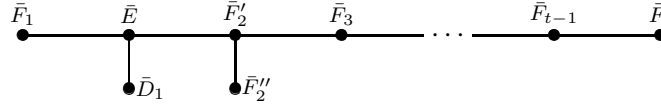
Dual graph of morphisms limiting to l_1

We can apply the induction hypothesis to these last morphisms to conclude that we may deform the morphism f to a morphism $m : \bar{F} := \bar{D}_1 \cup \bar{F}_1 \cup \dots \cup \bar{F}_t \rightarrow X$, where $m|_{\bar{F}_1 \cup \dots \cup \bar{F}_t}$ is a morphism in standard form and $m|_{\bar{D}_1}$ is birational onto its image and the image is a rational divisor in $|-K_X|$, or $|-rK_X|$ if $\deg X = 1$. We may specify which rational curve $m(\bar{D}_1)$ is if it represents $-K_{X_8}$, and we assume it is S if and only if \bar{F}_1 is not mapped to S . If \bar{F}_1 represents a divisor class different from $|-K_{X_8}|$, then we may assume that \bar{D}_1 is adjacent to \bar{F}_1 and conclude; if \bar{F}_1 represents the divisor class $|-K_{X_8}|$, then \bar{D}_1 is adjacent to \bar{F}_2 and \bar{D}_1 represents the divisor class $-rK_{X_8}$ for some $r \in \{1, 2, 3\}$. The divisor represented by \bar{F}_2 is either $-K_{X_\alpha}$, for some $\alpha \leq 7$ or it is a divisor on a del Pezzo surface of degree eight, i.e. $\mathbb{P}^1 \times \mathbb{P}^1$ or $Bl_p(\mathbb{P}^2)$.

We are only going to treat the case $m_*[\bar{F}_2] = -K_{X_7}$, and $m_*[\bar{D}_1] = -K_{X_8}$. The remaining cases are simpler and can be treated with similar techniques.

Let $J \subset \overline{\mathcal{M}}_{bir}(X, h_*[\bar{C}])$ be the closure of the set of morphisms $m' : \bar{D}'_1 \cup \bar{F}'_1 \cup \dots \cup \bar{F}'_t \longrightarrow X_8$ such that $m'|_{\bar{F}'_2 \cup \dots \cup \bar{F}'_t} \simeq m|_{\bar{F}_2 \cup \dots \cup \bar{F}_t}$, $m'|_{\bar{D}'_1} \simeq m|_{\bar{D}_1}$ the image of \bar{F}'_2 is a general rational divisor in $|-K_{X_7}|$ and the dual graphs of m and m' coincide.

We clearly have a morphism $J \rightarrow \overline{\mathcal{M}}_{bir}(X, -K_{X_7})$ obtained by “restricting a morphism in J to its \bar{F}'_2 component.” Since the intersection number $m_*[\bar{F}_2] \cdot m_*[\bar{D}_1]$ equals two, it follows that J has at most two irreducible components. Moreover, even if the space J is reducible, its components meet. To see this, we construct a point in common to the two components. Let $\psi : X_8 \longrightarrow \mathbb{P}^2$ be the morphism induced by the linear system $|-K_{X_7}|$. The morphism ψ contracts a (-1) -curve and ramifies above a smooth plane quartic R . The images of \bar{D}_1 and \bar{F}_1 are two distinct tangent lines in \mathbb{P}^2 which contain the image c of the contracted component and are tangent to R , but are not bitangent lines to R . The images of the curves \bar{F}'_2 are tangent lines to R . Since R has degree four, it follows that there is a point $c' \in R$ where the image of \bar{F}_1 meets transversely R . Through such a point c' , there are ten tangent lines to R , different from the tangent line to R at c' . Each of these lines corresponds to a point in the intersection of the two components of J . Moreover, these points in common to the components are easily seen to be smooth points of the mapping space, using Proposition 1.17. We deduce that J is connected, and thus in the same irreducible component of $\overline{\mathcal{M}}_{bir}(X, h_*[\bar{C}])$ containing m there is a morphism m_1 with dual graph

Dual graph of m_1

which agrees with m on the components with the same label and such that \bar{F}''_2 is mapped to the divisor class contracted by ψ and \bar{F}'_2 is mapped to a rational divisor in $|-K_{X_8}|$ distinct from the images of \bar{D}_1 and \bar{F}_1 . It follows from the computations in **Step 6, Case 4** below that m_1 represents a smooth point of the mapping space. We may now smooth $\bar{D}_1 \cup \bar{E} \cup \bar{F}_1$ to a single irreducible component \bar{G}_1 representing a nodal rational divisor in $|-2K_{X_8}|$. Similarly, we may smooth $\bar{F}'_2 \cup \bar{F}''_2$ to an irreducible component \bar{G}_2 representing a nodal rational divisor in $|-K_{X_7}|$. The resulting morphism m_2 has dual graph

Dual graph of m_2

This concludes the proof of this step.

We now define a locally closed subset K_β of $\overline{\mathcal{M}}_{0,0}(X, \beta)$. Let \bar{K}_β be the closure of the set of morphisms in standard form. The subspace $K_\beta \subset \bar{K}_\beta$ is the open subset of points lying in a unique irreducible component of $\overline{\mathcal{M}}_{0,0}(X, \beta)$, or equivalently K_β is the complement in \bar{K}_β of the union of all the pairwise intersections of the irreducible components of $\overline{\mathcal{M}}_{0,0}(X, \beta)$. In particular, all the points of \bar{K}_β that are smooth points in $\overline{\mathcal{M}}_{0,0}(X, \beta)$ lie in K_β .

Step 5. The morphisms in standard form are contained in K_β .

It is enough to prove that a morphism in standard form is a smooth point of $\overline{\mathcal{M}}_{0,0}(X, \beta)$. Let $k : \bar{K} \rightarrow X$ be a morphism in standard form and let $\bar{K}_1, \dots, \bar{K}_\ell$ be the components of \bar{K} . We will always assume that the numbering of the components is the “standard” one. The morphism k represents a smooth point of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ since $k^*\mathcal{T}_X$ is globally generated on all components \bar{K}_i , for $i \geq 2$ and $H^1(\bar{K}_1, k^*\mathcal{T}_X) = 0$.

Step 6. The space K_β is connected.

To prove connectedness of K_β , let $k : \bar{K} \rightarrow X$ be a morphism in standard form and suppose that all the nodes of \bar{K} are mapped to points of X not lying on (-1) -curves. There are such morphisms in all the connected components of K_β since $k|_{\bar{K}_i}$ is a free morphism, for $i \geq 2$ and S is not a (-1) -curve. Given any $k' : \bar{K}' \rightarrow X$, we construct a deformation from k' to k entirely contained in K_β . This is clearly enough to prove the connectedness of K_β .

We are going to construct the deformation in stages.

We prove that in the same connected component of K_β containing k' there is a morphism $k_1 : \bar{K}_1^1 \cup \dots \cup \bar{K}_\ell^1 \rightarrow X$ such that $k_1|_{\bar{K}_1^1} \simeq k|_{\bar{K}_1}$.

This is true by assumption if $k_*[\bar{K}_1] = -K_{X_8}$, since in this case $k(\bar{K}_1) = S = k'(\bar{K}_1')$ and k and k' are birational. Thus in this case we may choose $k_1 = k'$.

Suppose that $k_*[\bar{K}_1] \neq -K_{X_8}$. Since $k'|_{\bar{K}_i}$ is free for all i 's, we may assume that $k'(\bar{K}_i)$ is not contained in the image of k , for all i 's. Thanks to Theorem 2.12, Theorem 2.10 and Theorem 3.3, we conclude that there is an irreducible curve $P \subset K_{k_*[\bar{K}_1]} \subset \overline{\mathcal{M}}_{0,0}(X, k_*[\bar{K}_1])$ containing $k'|_{\bar{K}_1'}$ and $k|_{\bar{K}_1}$. Consider the morphism

$$\mathrm{Sl}_{k'}(\bar{K}_1') \xrightarrow{\pi} \overline{\mathcal{M}}_{0,0}(X, k_*[\bar{K}_1])$$

and let $\bar{P} \subset \pi^{-1}(P)$ be an irreducible curve dominating P and containing k' . The curve \bar{P} has finitely many points not lying in K_β : they are the points \tilde{k} for which the image of $\pi(\tilde{k})$ contains a component of $k'(\bar{K}_2' \cup \dots \cup \bar{K}_\ell')$. By

construction, \bar{P} contains a morphism $k_1 : \bar{K}_1^1 \cup \dots \cup \bar{K}_\ell^1 \longrightarrow X$ such that $k_1|_{\bar{K}_1^1} \simeq k|_{\bar{K}_1}$ and $k_1|_{\bar{K}_2^1 \cup \dots \cup \bar{K}_\ell^1} \simeq k'|_{\bar{K}_2' \cup \dots \cup \bar{K}_\ell'}$. It follows that $\bar{P} \cap K_\beta$ is an irreducible curve contained in K_β and containing k' and k_1 . Therefore k' and k_1 are in the same connected (in fact irreducible) component of K_β .

Thus to prove that K_β is connected we may assume that $k'|_{\bar{K}_1'} \simeq k|_{\bar{K}_1}$. Suppose that we found a morphism k_{j-1} in the same connected component of K_β as k' such that $k_{j-1}|_{\bar{K}_1^{j-1} \cup \dots \cup \bar{K}_{j-1}^{j-1}} \simeq k|_{\bar{K}_1 \cup \dots \cup \bar{K}_{j-1}}$ for some $2 \leq j \leq \ell$. If we can find a connected subset of K_β containing k_{j-1} and a morphism $k_j : \bar{K}_1^j \cup \dots \cup \bar{K}_\ell^j \longrightarrow X$ such that $k_j|_{\bar{K}_1^j \cup \dots \cup \bar{K}_j^j} \simeq k|_{\bar{K}_1 \cup \dots \cup \bar{K}_j}$, then we may conclude by induction on j .

The remaining part of the proof will examine the several cases separately. To simplify the notation we assume that $j = 2$, we write \bar{K}_1 also for \bar{K}_1^1 since $k|_{\bar{K}_1} \simeq k_2|_{\bar{K}_1^1}$, and we let $k = k|_{\bar{K}_1 \cup \bar{K}_2}$ and $k_1 = k_1|_{\bar{K}_1 \cup \bar{K}_2^1}$; to get the result for k and k_1 simply consider the morphism

$$\mathrm{Sl}_{k_1}(\bar{K}_2^1) \xrightarrow{\pi} \overline{\mathcal{M}}_{0,0}(X, k_*[\bar{K}_2])$$

and lift the path in $\overline{\mathcal{M}}_{0,0}(X, k_*[\bar{K}_2])$ to a path in $\overline{\mathcal{M}}_{0,0}(X, k_*[\bar{K}])$, and note that the lift lies in the space K_β .

Case 1: $k_*[\bar{K}_2]$ is a multiple of a conic. We may assume that the node between \bar{K}_1 and \bar{K}_2^1 is mapped to the same point where the node between \bar{K}_1 and \bar{K}_2 is mapped; denote this point by p_2 . It follows that the image of \bar{K}_2 is uniquely determined. From the irreducibility of the Hurwitz spaces ([Fu2]) it follows that we may find an irreducible curve in K_β containing k_1 and a morphism k_2 as above.

Case 2: $k_*[\bar{K}_2] = -K_{X_\alpha}$, for $1 \leq \alpha \leq 6$. We may assume that the node between \bar{K}_1 and \bar{K}_2^1 is mapped to the same point where the node between \bar{K}_1 and \bar{K}_2 is mapped; denote this point by p_2 .

Since the point p_2 does not lie on any (-1) -curve, the space of all rational divisors in $|-K_{X_\alpha}|$ containing the point p_2 is isomorphic to the space of all rational divisors in $|-K_{\tilde{X}_\alpha}|$, where \tilde{X}_α is the blow up of X_α at p_2 . It follows from the fact that \tilde{X}_α is a del Pezzo surface of degree at least two, that the space of rational curves in $|-K_{\tilde{X}_\alpha}|$ is irreducible and thus we conclude also in this case.

Case 3: $k_*[\bar{K}_2] = -K_{X_7}$ and $k_*[\bar{K}_1] = -K_{X_7}$ or $-K_{X_8}$. The dual graphs of k and k_1 are

$$\begin{array}{cc} \bar{K}_1 & \bar{K}_2 \\ \bullet & \bullet \\ \hline & \end{array} \quad \begin{array}{cc} \bar{K}_1 & \bar{K}_2^1 \\ \bullet & \bullet \\ \hline & \end{array}$$

Dual graphs of k and k_1

and we have $k_1(\bar{K}_1) \cdot k_1(\bar{K}_2^1) = 2$. Consider the diagram

$$\begin{array}{ccccc}
 S_{bir} & \xrightarrow{\quad} & S_2 & \xrightarrow{\quad} & \bar{K}_1 \times \bar{K}_1 \\
 \downarrow F & & \downarrow & & \downarrow (k|_{\bar{K}_1}, k|_{\bar{K}_1}) \\
 & & \overline{\mathcal{M}}_{0,2}(X, -K_{X_7}) & \xrightarrow{ev} & X \times X \\
 & & \downarrow & & \\
 \overline{\mathcal{M}}_{bir}(X, -K_{X_7}) & \longrightarrow & \overline{\mathcal{M}}_{0,0}(X, -K_{X_7}) & &
 \end{array}$$

where all squares are fiber products. The morphism F is dominant and its fiber over a stable map f has length two, unless the image of f contains $k(\bar{K}_1)$. We denote by $S'_{bir} \subset S_{bir}$ the union of the components of S_{bir} dominating $\overline{\mathcal{M}}_{bir}(X, -K_{X_7})$. Clearly k and k_1 both lie in S'_{bir} . Moreover, since the fibers of $F' := F|_{S'_{bir}}$ have length two, it follows that S'_{bir} has at most two components. To prove the connectedness of S'_{bir} , we assume it is reducible and check that there is a point in K_β common to the two components of S'_{bir} . This will conclude the proof in this case.

Consider the morphism $\varphi : X \rightarrow \mathbb{P}^2$ determined by $|-K_{X_7}|$. We can factor φ as the contraction of the divisor class e_8 followed by the double cover of \mathbb{P}^2 branched along a smooth plane quartic curve R . The image of \bar{K}_2^1 is a line tangent to R . The image of the component \bar{K}_1 is herself a tangent line to the branch curve R . Note that in case $k_*[\bar{K}_1] = -K_{X_7}$ we may assume that this tangent line is not a bitangent line nor a flex line. In case $k_*[\bar{K}_1] = -K_{X_8}$, it follows from the fact that all the rational divisors in $|-K_{X_8}|$ are nodal that the image of \bar{K}_1 is not a flex line; the fact that it is not a bitangent line follows from the fact that X is a del Pezzo surface.

Let $s \in \mathbb{P}^2$ be one of the two points such that $s \in R \cap \varphi(k(\bar{K}_1))$, but s is not the point where R and $\varphi(k(\bar{K}_1))$ are tangent. Through the point s there are ten tangent lines to R (counted with multiplicity, and not counting the tangent line to R at s): tangent lines through s correspond to ramification points of the morphism $R \rightarrow \mathbb{P}^1$ obtained by projecting away from the point s . Since R has genus three and the morphism has degree three, by the Hurwitz formula we deduce that the degree of the ramification divisor is ten, as asserted above. Let $L \subset \mathbb{P}^2$ be one of the tangent lines to R through s different from $\varphi(k(\bar{K}_1))$ and let $f : \mathbb{P}^1 \rightarrow X$ be a morphism birational to

its image and whose image is $\varphi^{-1}(L)$. The morphism f represents a point of $\overline{\mathcal{M}}_{bir}(X, -K_{X_7})$ above which the two components of S'_{bir} must meet. Such a point is smooth thanks to Proposition 1.17. This concludes the proof in this case.

Case 4: $k_*[\bar{K}_2] = -K_{X_7}$ and $k_*[\bar{K}_2] = -2K_{X_8}$. The dual graphs of k and k_1 are

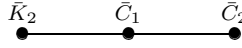
Dual graphs of k and k_1

We reduce this case to the previous one with the following construction. We deform k and k_1 inside K_β to morphisms k' and k'_1 respectively with dual graphs

Dual graphs of k' and k'_1

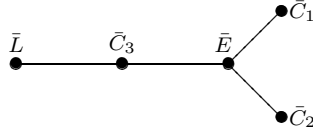
where \bar{C}_1 and \bar{C}_2 are mapped to two given distinct rational divisors M_1 and M_2 in $|-K_{X_8}|$.

The strategy is the same for k and for k_1 , therefore we will only describe the deformation for k . We may deform $k|_{\bar{K}_1}$ to the morphism $k'|_{\bar{C}_1 \cup \bar{C}_2}$, thanks to the irreducibility of $\overline{\mathcal{M}}_{bir}(X, -2K_{X_8})$. This means we may deform k to a morphism \bar{k} which is either k' or it has dual graph

Dual graph of \bar{k}

Since $(-K_{X_8}) \cdot (-K_{X_7}) = 2$, there are at most two irreducible components of the space of morphisms with dual graph as above. Thanks to the previous case, we know that this space is connected.

Let $\tilde{k}' : \bar{L} \cup \bar{C}_3 \rightarrow X$ be a stable map birational to its image, where \bar{L} is mapped to the (-1) -curve with divisor class e_8 , \bar{C}_3 is mapped to a rational divisor in $|-K_{X_8}|$ different from the images of both \bar{C}_1 and \bar{C}_2 . By the connectedness established above, the (closure of the) same connected component of K_β containing \bar{k} contains a morphism $\tilde{k} : \hat{C} \rightarrow X$ with dual graph



Dual graph of \tilde{k}

where \bar{E} is contracted to the base-point of $|-K_{X_8}|$. To check that \tilde{k} is in K_β it is enough to check that \tilde{k} represents a smooth point of $\overline{\mathcal{M}}_{0,0}(X, k_*[\bar{K}])$.

The point represented by $\tilde{k} : \tilde{C} \rightarrow X$ in $\overline{\mathcal{M}}_{0,0}(X, k_*[\bar{K}])$ is smooth if $H^0(\tilde{C}, \mathcal{C}_{\tilde{k}}) = 0$ (we are using the notation of (1.3)). We have a natural inclusion

$$H^0(\tilde{C}, \mathcal{C}_{\tilde{k}}) \subset H^0(\tilde{C}, \tilde{k}^* \Omega_X^1 \otimes \omega_{\tilde{C}})$$

We prove first that any global section of $\mathcal{C}_{\tilde{k}}$ is zero on $\bar{L} \cup \bar{C}_3$ and then that any global section of $\tilde{k}^* \Omega_X^1 \otimes \omega_{\tilde{C}}$ vanishing on $\bar{L} \cup \bar{C}_3$ is the zero section.

The first assertion is clear from Proposition 1.18: there are no non-zero global sections of $\mathcal{C}_{\tilde{k}}$ on \bar{L} , and since the sheaf $\mathcal{C}_{\tilde{k}}$ is locally free near the node between \bar{L} and \bar{C}_3 , it follows that a global section of $\mathcal{C}_{\tilde{k}}$ must vanish at the node. Since the degree of the sheaf $\mathcal{C}_{\tilde{k}}$ on \bar{C}_3 is zero, it follows that a global section of $\mathcal{C}_{\tilde{k}}$ must vanish on $\bar{L} \cup \bar{C}_3$.

The second assertion is a consequence of the fact that $h^0(\tilde{C}, \tilde{k}^* \Omega_X^1 \otimes \omega_{\tilde{C}}) = 1$, and that a non-zero section of the sheaf $\tilde{k}^* \Omega_X^1 \otimes \omega_{\tilde{C}}$ is not identically zero on \bar{C}_3 .

To compute $h^0(\tilde{C}, \tilde{k}^* \Omega_X^1 \otimes \omega_{\tilde{C}})$, we use Serre duality to deduce that

$$h^0(\tilde{C}, \tilde{k}^* \Omega_X^1 \otimes \omega_{\tilde{C}}) = h^1(\tilde{C}, \tilde{k}^* \mathcal{T}_X)$$

There is a short exact sequence of sheaves

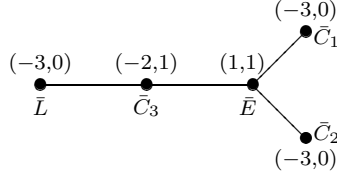
$$0 \longrightarrow \tilde{k}^* \mathcal{T}_X \longrightarrow k_L^* \mathcal{T}_X \oplus k_E^* \mathcal{T}_X \oplus \bigoplus_i k_i^* \mathcal{T}_X \longrightarrow \bigoplus_{\substack{\nu \text{ a node} \\ \text{of } \tilde{C}}} \mathcal{T}_{X, \tilde{k}(\nu)} \longrightarrow 0$$

Note that the restriction of $\tilde{k}^* \mathcal{T}_X$ to every non contracted component is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, where the subsheaf of degree two is canonically the tangent sheaf of the component. The associated long exact sequence to the sequence above is

$$0 \longrightarrow k^{h^0} \longrightarrow k^{14} \longrightarrow k^8 \longrightarrow k^{h^1} \longrightarrow 0$$

Since the images of all the non-contracted components are pairwise transverse (all the intersection numbers are one), and since the only global sections come from the tangent vector fields, it follows that any global section must vanish at all nodes. Thus, there are two global sections coming from each of the curves \bar{L} , \bar{C}_1 and \bar{C}_2 and only one coming from \bar{C}_3 . We deduce that $h^0 = 7$ and finally $h^1 = 1$, as asserted above.

Let us go back to the sheaf $\tilde{k}^*\Omega_X^1 \otimes \omega_{\tilde{C}}$. We just computed that this sheaf has exactly one global section. We have the following decomposition for the degrees of the restrictions of the sheaf $\tilde{k}^*\Omega_X^1 \otimes \omega_{\tilde{C}}$ to each component:



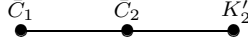
Degrees of the sheaf $\tilde{k}^*\Omega_X^1 \otimes \omega_{\tilde{C}}$

where the pair of numbers next to a vertex represent the degrees of $\tilde{k}^*\Omega_X^1 \otimes \omega_{\tilde{C}}$ restricted to the component represented by the corresponding vertex. We examine the vertex of valence three in the dual graph. Necessary conditions for a section of $\tilde{k}^*\Omega_X^1 \otimes \omega_{\tilde{C}}$ on \bar{E} to extend to a global section are that the section “points in the right direction” at the nodes. These are clearly linear conditions and there are three such conditions. Moreover, every section satisfying the stated conditions extends uniquely to a global section: this is obvious on the components \bar{C}_1 and \bar{C}_2 . For the remaining components, note that every global section must vanish at the node between \bar{L} and \bar{C}_3 , since the intersection number $\tilde{k}(\bar{L}) \cdot \tilde{k}(\bar{C}_3)$ equals one, and therefore the intersection is transverse. Thus every global section of $\tilde{k}^*\Omega_X^1 \otimes \omega_{\tilde{C}}$ is uniquely determined by its restriction to \bar{E} . Thus the only way a section can be identically zero on \bar{C}_3 is if the sections on \bar{E} all vanish at the node $\bar{C}_3 \cap \bar{E}$. Note that the three tangent directions of the images of \bar{C}_1 , \bar{C}_2 and \bar{C}_3 at their common point p are pairwise independent. Choose homogeneous coordinates E_1, E_2 on \bar{E} such that $[0, 1] = \bar{E} \cap C_1$, $[1, 0] = \bar{E} \cap C_2$. Choose local coordinates u, v on X near p such that the zero set of u is tangent to the image of \bar{C}_1 and the zero set of v is tangent to the image of \bar{C}_2 . Rescaling by a non-zero constant u and v we may also assume that the zero set of $u + v$ is tangent to the image of \bar{C}_3 . The restrictions of the global sections of $\tilde{k}^*\Omega_X^1 \otimes \omega_{\tilde{C}}$ to \bar{E} are multiples of the section

$$\sigma := E_0 du + E_1 dv$$

In particular, if a section vanishes at one of the nodes between \bar{E} and \bar{C}_i , then it vanishes identically. This concludes the proof that the sheaf $\mathcal{C}_{\tilde{k}}$ has no global sections and thus we conclude that \tilde{k} is a smooth point of $\overline{\mathcal{M}}_{0,0}(X, k_*[\bar{K}])$.

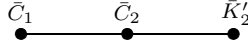
We now resume our argument. It is clear that \tilde{k} is also a limit of morphisms \tilde{k}' with dual graph



Dual graph of the morphisms \tilde{k}'

which is precisely what we wanted to prove. This completes the reduction of this case to **Case 3**, and thus this case is proved.

Case 5: $k_*[\bar{K}_2] = -K_{X_7}$ and $k_*[\bar{K}_1] = -3K_{X_8}$. We also reduce this case to **Case 3**. As before, thanks to the irreducibility of $\overline{\mathcal{M}}_{bir}(X, -3K_{X_8})$ we may deform the morphism k so that $k|_{\bar{K}_1}$ is replaced by the birational morphism $k' : \bar{C}_1 \cup \bar{C}_2 \rightarrow X$, where \bar{C}_1 is immersed and represents $|-2K_{X_8}|$, and \bar{C}_2 is mapped to a given rational divisor in $|-K_{X_8}|$. After possibly sliding the component \bar{K}_2 along \bar{C}_1 , we may suppose that the dual graph of k' is the following:



Dual graph of k'

Similar remarks apply to k_2 . This completes the reduction to **Case 3** and the proof in this case.

Case 6: $k_*[\bar{K}_2] = -2K_{X_8}$ and $k_*[\bar{K}_1] = -2K_{X_8}$. Since the intersection product $(-2K_{X_8})^2$ equals four, and the space $\overline{\mathcal{M}}_{bir}(X, -2K_{X_8})$ is irreducible, it follows that there are at most four irreducible components of morphisms in standard form representing the divisor class $-4K_{X_8}$. Let $c : \bar{C}_1 \cup \bar{C}_2 \rightarrow X$ be a stable map birational to its image such that \bar{C}_i is mapped to a (-1) -curve C_i and $C_1 + C_2 = -2K_{X_8}$. Consider the morphism

$$\mathrm{Sl}_k(\bar{K}_1) \xrightarrow{\pi} \overline{\mathcal{M}}_{bir}(X, -2K_{X_8})$$

The morphism π is dominant. Thus we may find a morphism $k' : \bar{C}_1 \cup \bar{C}_2 \cup \bar{K}_2 \rightarrow X$ such that $k'|_{\bar{C}_1 \cup \bar{C}_2} \simeq c$, lying in the same irreducible component of K_β as k . We have two possibilities for the dual graph of k' :

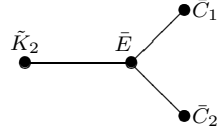


Possible dual graphs of k'

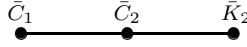
We want to reduce to the case in which \bar{K}_2 is adjacent to \bar{C}_2 . Consider the morphism

$$\mathrm{Sl}_{k'}(\bar{K}_2) \xrightarrow{a} \bar{C}_1$$

and as usual this morphism is dominant. This means that we may slide the node between \bar{K}_2 and \bar{C}_1 until it reaches the node between \bar{C}_1 and \bar{C}_2 . The resulting morphism \tilde{k} has dual graph

Dual graph of \tilde{k}

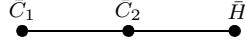
where \bar{E} is contracted by \tilde{k} . It is easy to check that this morphism represents a smooth point of $\overline{\mathcal{M}}_{0,0}(X, -4K_{X_8})$ and that it is also a limit of morphisms \tilde{k}' with dual graph

Dual graph of \tilde{k}'

Thus we may indeed assume that \bar{K}_2 is adjacent to \bar{C}_2 . Note that since $-2K_{X_8} \cdot C_1 = 2$, it follows that there are at most two connected components in the space of morphisms in standard form representing the divisor class $-4K_{X_8}$. To conclude, it is enough to show that we may “exchange” the two intersection points $C_2 \cap k'(\bar{K}_2)$ by a connected path contained in K_β .

Consider the morphism $\varphi : X \rightarrow \mathbb{P}^3$ induced by the linear system $|-2K_{X_8}|$. We have already seen that the image is a quadric cone Q and that the morphism is ramified along a smooth curve R which is the complete intersection of Q with a cubic surface. The (-1) -curves C_1 and C_2 have as image the intersection of Q with a plane which is everywhere tangent to the curve R (and does not contain the vertex of the cone). Let p be one of the intersection points of $\varphi(C_2)$ with R . Projection away from the tangent line L to R at p determines a morphism $\pi_L : R \rightarrow \mathbb{P}^1$ of degree four. Since the genus of R is four, it follows from the Hurwitz formula that the degree of the ramification divisor of π_L is 14. It is immediate to check that π_L ramifies above the tangent plane to Q at p , and that the ramification index is two. It is also immediate that above the plane containing $\varphi(C_2)$ the ramification index is two. We deduce that there are planes in the pencil containing L which are tangent to R and are not the tangent plane to Q at p nor the plane containing $\varphi(C_2)$. Such planes correspond to rational divisors H in $|-2K_{X_8}|$ with the property that $H \cap C_2$ consists of the unique point $\varphi^{-1}(p)$. Let $\nu : \mathbb{P}^1 \rightarrow X$ be a birational morphism whose image is one of the divisors H constructed above. The morphism ν represents a morphism in $\overline{\mathcal{M}}_{bir}(X, -2K_{X_8})$, and since this space is irreducible, we may deform \tilde{k}'

to a morphism $\bar{k} : \bar{C}_1 \cup \bar{C}_2 \cup \bar{H} \longrightarrow X$ with dual graph



Dual graph of \bar{k}

and such that $\bar{k}|_{\bar{H}} \simeq \nu$. The morphism \bar{k} represents a smooth point of the space $\overline{\mathcal{M}}_{0,0}(X, -4K_{X_8})$, thanks to Proposition 1.17. Thus $\bar{k} \in K_\beta$ and it lies in the same connected component of K_β as k .

Applying the same construction to the morphism k_2 , we obtain that also k_2 lies in the same connected component of K_β as \bar{k} . This concludes the proof of this case.

Case 7: $k_*[\bar{K}_2] = -2K_{X_8}$ and $k_*[\bar{K}_1] = -3K_{X_8}$. Let $c : \bar{C}_1 \cup \bar{C}_2 \longrightarrow X$ be a morphism birational to its image, such that $c(\bar{C}_2)$ is a rational divisor in $|-K_{X_8}|$ and $c(\bar{C}_1)$ is a general rational divisor in $|-2K_{X_8}|$. Consider the morphism

$$\mathrm{Sl}_k(\bar{K}_1) \xrightarrow{\pi} \overline{\mathcal{M}}_{bir}(X, -3K_{X_8})$$

and note that as usual it is dominant. Therefore we may deform k to a morphism $k' : \bar{C}_1 \cup \bar{C}_2 \cup \bar{K}_2 \longrightarrow X$ such that $k'|_{\bar{C}_1 \cup \bar{C}_2} \simeq c$.

As before, we may slide the component \bar{K}_2 along \bar{C}_1 until it reaches \bar{C}_2 , and reduce to the case in which \bar{K}_2 is adjacent to \bar{C}_2 . The same considerations of the final step of the previous case allow us to conclude.

This concludes the proof of the connectedness of K_β .

Step 7. We now simply collect all the information we obtained, to conclude the proof of the theorem. **Step 4** and **Step 5** imply (under the hypotheses of the theorem) that every irreducible component of $\overline{\mathcal{M}}_{bir}(X, \beta)$ is either empty or it contains a point lying in K_β . **Step 6** then implies that there is at most one component of $\overline{\mathcal{M}}_{bir}(X, \beta)$ containing K_β . Thus if $\overline{\mathcal{M}}_{bir}(X, \beta)$ is not empty, then it consists of exactly one irreducible component. This concludes the proof of the theorem. \square

Proposition 4.6 *Let X be a del Pezzo surface. If β is a nef divisor which is not a multiple of a conic, then the space $\overline{\mathcal{M}}_{bir}(X, \beta)$ is not empty.*

Proof. We may write

$$\beta = n_8(-K_{X_8}) + \dots + n_2(-K_{X_2}) + \beta'$$

where $n_8, \dots, n_2 \geq 0$ and β' is a nef divisor on a del Pezzo surface X_1 of degree eight dominated by X (if $X \simeq \mathbb{P}^2$ the assertion is obvious).

If $n_8 \geq 2$, then we define n and r by the conditions $n_8 = 2(n-1) + r$, where $r = 2, 3$; if $n_8 = 1$, then we define $n = r = 1$; if $n_8 = 0$, then we define $n = r = 0$. Let

$$\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}\ell_1 \oplus \mathbb{Z}\ell_2 \quad \text{where } \ell_1 = \{p\} \times \mathbb{P}^1, \ell_2 = \mathbb{P}^1 \times \{p\}$$

$$\text{Pic}(Bl_p(\mathbb{P}^2)) \simeq \mathbb{Z}\ell \oplus \mathbb{Z}e \quad \text{where } \ell^2 = 1, \ell \cdot e = 0, e^2 = -1$$

and write

$$\beta' = n_1(\ell_1 + \ell_2) + n_0\ell_2 \quad \text{if } X \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

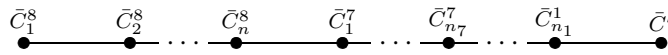
$$\beta' = n_1\ell + n_0(\ell - e) \quad \text{if } X \simeq Bl_p(\mathbb{P}^2)$$

where $n_1 \geq 0, n_0 \geq 0$ (we may need to exchange ℓ_1, ℓ_2). Note that with this notation the divisor β is multiple of a conic if and only if $n_8 = n_7 = \dots = n_1 = 0$.

Choose

- $n-1$ distinct rational integral nodal divisors C_2^8, \dots, C_n^8 in $|-2K_{X_8}|$;
- a rational integral nodal divisor C_1^8 (different from the previous ones if $r = 2$) in $|-rK_{X_8}|$;
- n_i distinct rational integral nodal divisors $C_1^i, \dots, C_{n_i}^i$ in $|-K_{X_i}|$;
- n_1 distinct integral divisors C_1, \dots, C_{n_1} lying in $|\ell_1 + \ell_2|$, if $X_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and lying in $|\ell|$, if $X_1 \simeq Bl_p(\mathbb{P}^2)$;
- an integral divisor C' in $|\ell_2|$ or $|\ell - e|$.

Having made these choices, we may now consider the stable map of genus zero $f : \bar{C} \rightarrow X$, with dual graph



Dual graph of \bar{f}

where of course we ignore a component if the corresponding curve without a bar has not been defined. The morphism f on a component \bar{D} is the normalization of the curve D followed by inclusion in X , if $D \neq C'$, and it is a multiple cover of degree n_0 , if $D = C'$.

All the restrictions of f to the irreducible components of \bar{C} different from \bar{C}_1^8 are free morphisms; the cohomology group $H^1(\bar{C}_1^8, f^*\mathcal{T}_X)$ is immediately seen to be zero. Thus we may deform f to a morphism lying in $\mathcal{M}_{0,0}(X, \beta)$. If the general deformation of f were a morphism not birational to its image,

then $f_*[\bar{C}]$ would not be reduced. Since this is not the case, it follows that we may deform f to a morphism with irreducible domain, which is birational to its image. This proves that $\mathcal{M}_{bir}(X, \beta) \neq \emptyset$, if β is not a multiple of a conic. This concludes the proof of the proposition. \square

Remark 1. The spaces $\mathcal{M}_{0,0}(X, mC)$, where C is the class of a conic, are easily seen to be irreducible, for $m \geq 1$. If $m = 1$, we have $\overline{\mathcal{M}}_{bir}(X, C) \simeq \mathbb{P}^1$. If $m \geq 1$, then there is a morphism $\mathcal{M}_{0,0}(X, mC) \rightarrow \mathcal{M}_{0,0}(X, C)$, obtained by “forgetting the multiple cover.” The fibers of this morphism are birational to Hurwitz schemes, which are irreducible ([Fu2]). The irreducibility of $\mathcal{M}_{0,0}(X, mC)$ follows.

Remark 2. If L is an integral divisor of anticanonical degree one, then either L is a (-1) -curve, or it is the anticanonical divisor on a del Pezzo surface of degree one.

If L is a (-1) -curve, the space $\overline{\mathcal{M}}_{0,0}(X, L)$ has dimension zero and length one; it therefore consists of a single reduced point and is irreducible.

If $L = -K_X$, the three spaces $\overline{\mathcal{M}}_{0,0}(X, -K_X)$, $\mathcal{M}_{0,0}(X, -K_X)$ and $\overline{\mathcal{M}}_{bir}(X, -K_X)$ are all equal and have dimension zero and length twelve. They are not irreducible. For a general del Pezzo surface of degree one, the space $\overline{\mathcal{M}}_{bir}(X, -K_X)$ is reduced and consists of exactly twelve points. This happens precisely when the rational divisors in $|-K_X|$ are all nodal.

5 Divisors of Small Degree on X_8

5.1 The Divisor $-K_{X_8} - K_{X_7}$

Here we prove the irreducibility of the spaces $\overline{\mathcal{M}}_{bir}(X, \beta)$, where the degree of X is one, β is ample and the anticanonical degree of β is three. We already saw (Theorem 2.12) that the space $\overline{\mathcal{M}}_{bir}(X, -3K_X)$ is irreducible. The following proofs are similar to the proof of Theorem 2.12.

Lemma 5.1 *Let X be a del Pezzo surface of degree one. Suppose that all the rational divisors in $|-K_X|$ are nodal and that $\overline{\mathcal{M}}_{bir}(X, -2K_X)$ is irreducible. Let $L \subset X$ be a (-1) -curve and let $b : X \rightarrow X'$ be the contraction of L . Then the space $\overline{\mathcal{M}}_{bir}(X, -K_X - K_{X'})$ is irreducible.*

Proof. Let $f : \mathbb{P}^1 \rightarrow X$ be a morphism in $\overline{\mathcal{M}}_{bir}(X, -K_X - K_{X'})$ and suppose that the image of f contains the independent point p . Consider the space of morphisms of $\overline{\mathcal{M}}_{bir}(X, -K_X - K_{X'})$ in the same irreducible component as $[f]$ which contain the point p in their image, denote this space by $\overline{\mathcal{M}}_{bir}(p)$. It follows immediately from the dimension estimates (1.14)

that $\dim_{[f]} \overline{\mathcal{M}}_{bir}(p) = 1$ and that $[f]$ is a smooth point of $\overline{\mathcal{M}}_{bir}(p)$. We may therefore find a smooth irreducible projective curve B , a normal surface $\pi : S \rightarrow B$ and a morphism $F : S \rightarrow X$ such that the induced morphism $B \rightarrow \overline{\mathcal{M}}_{bir}(p)$ is surjective onto the component containing $[f]$. From [Ko] Corollary II.3.5.4, it follows immediately that the morphism F is dominant. We want to show that there are reducible fibers of π . The argument is the same that appears at the beginning of the proof of Theorem 2.12.

Thus there must be a morphism $f_0 : \bar{C} \rightarrow X$ with reducible domain in the family of stable maps parametrized by B , and since all such morphisms contain the general point p in their image, the same is true of the morphism f_0 . In particular, since the point p does not lie on any rational curve of anticanonical degree one, it follows that \bar{C} consists of exactly two components \bar{C}_1 and \bar{C}_2 , where each \bar{C}_i is irreducible and we may assume that $f_0(\bar{C}_1)$ has anticanonical degree one and $f_0(\bar{C}_2)$ has anticanonical degree two. Denote by C_i the image of \bar{C}_i . It also follows from the definition of an independent point and Proposition 1.17 that f_0 represents a smooth point of $\overline{\mathcal{M}}_{bir}(X, -K_X - K_{X'})$.

There are two possibilities for C_1 : either it is a (-1) -curve or it is a rational divisor in the anticanonical system. We want to prove that we may assume that C_1 is not a (-1) -curve.

Suppose C_1 is a (-1) -curve. The morphism $f_0|_{\bar{C}_2}$ is a free morphism, because the image contains a general point and has anticanonical degree two. Moreover the image C_2 , being a curve of anticanonical degree two, is one of the following: a conic, the anticanonical divisor on a del Pezzo surface of degree two dominated by X or a divisor in $|-2K_X|$. In all these cases we know that the space $\overline{\mathcal{M}}_{bir}(X, (f_0)_*[\bar{C}_2])$ is irreducible. Thus we may deform $f_0|_{\bar{C}_2}$ to a curve with two irreducible components, both mapped to (-1) -curves. Considering the space $\mathrm{Sl}_{f_0}(\bar{C}_2)$ we conclude that we may deform f_0 to a morphism $f_1 : \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \rightarrow X$ where each component \bar{L}_i is mapped to a different (-1) -curve L_i on X .

We deduce that we have $L_1 + L_2 + L_3 = -2K_X + L$ and L_1, L_2 and L_3 are distinct (-1) -curves. Thanks to Lemma 2.18 we conclude that there is a standard basis $\{\ell, e_1, \dots, e_8\}$ of $\mathrm{Pic}(X)$ such that

$$\begin{cases} L_1 &= -2K_X - e_1 \\ L_2 &= e_8 \\ L_3 &= e_1 \end{cases}, \quad \begin{cases} L_1 &= -2K_X - (\ell - e_7 - e_8) \\ L_2 &= \ell - e_7 - e_8 \\ L_3 &= e_8 \end{cases} \quad (5.22)$$

or

$$\begin{cases} L_1 &= -2K_X - (\ell - e_7 - e_8) \\ L_2 &= e_1 \\ L_3 &= \ell - e_1 - e_7 \end{cases} \quad (5.23)$$

after possibly permuting the indices 1, 2 and 3.

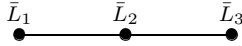
The next step in the deformation is to produce a component mapped to the divisor class $-2K_X$.

In the first case of (5.22), the component \bar{L}_1 is adjacent to both \bar{L}_2 and \bar{L}_3 , since $L_2 \cdot L_3 = 0$. We may therefore consider $\mathrm{Sl}_{f_1}(\bar{L}_1 \cup \bar{L}_3)$ to smooth $\bar{L}_1 \cup \bar{L}_3$ to a single component \bar{K} mapped to $-2K_X$.

In the second case of (5.22), either \bar{L}_1 and \bar{L}_2 are adjacent and it is enough to smooth their union to conclude, or \bar{L}_2 is adjacent to \bar{L}_3 and not to \bar{L}_1 . If this happens, then we may smooth the union $\bar{L}_2 \cup \bar{L}_3$ to a single irreducible component \bar{Q} , mapped to the conic $\ell - e_7$. Denote the resulting morphism by f'_1 . We may consider the dominant morphism

$$a : \mathrm{Sl}_{f'_1}(\bar{Q}) \longrightarrow \bar{L}_1$$

and let $\bar{e} \in \bar{L}_1$ be a point mapped to a point lying on the (-1) -curve with divisor class $L_2 = \ell - e_7 - e_8$. Since a is dominant, we may find a morphism f''_1 such that $a(f''_1) = \bar{e}$. By construction, the dual graph of the morphism f''_1 is



Dual graph of f''_1

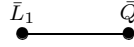
and we may now smooth $\bar{L}_1 \cup \bar{L}_2$ to conclude.

In the case of (5.23), we first prove that we may assume that \bar{L}_2 and \bar{L}_3 are adjacent. If \bar{L}_2 and \bar{L}_3 are not adjacent, then \bar{L}_1 is adjacent to both \bar{L}_2 and \bar{L}_3 and we may consider $\mathrm{Sl}_{f_1}(\bar{L}_1 \cup \bar{L}_2)$ to smooth $\bar{L}_1 \cup \bar{L}_2$ to a single irreducible component \bar{K} mapped to a curve with divisor class $K := (5; 1, 2, 2, 2, 2, 1, 1)$. Note that the divisor class of K is the divisor class of the anticanonical divisor on a del Pezzo surface of degree two dominated by X . Thus we know that the space $\overline{\mathcal{M}}_{bir}(X, K)$ is irreducible and it contains a point whose image consists of the union of the two (-1) -curves with divisor classes $L'_1 := (5; 1, 2, 2, 2, 2, 1, 2)$ and $L'_2 := (0; 0, 0, 0, 0, 0, 0, -1)$. Considering $\mathrm{Sl}_{f_1}(\bar{L}_1 \cup \bar{L}_2)$ we may therefore deform f_1 to a morphism $f_2 : \bar{L}'_1 \cup \bar{L}'_2 \cup \bar{L}_3 \rightarrow X$ such that the image of \bar{L}'_i is the (-1) -curve L'_i . Thus we have

$$\begin{cases} L'_1 = (5; 1, 2, 2, 2, 2, 2, 1, 2) \\ L'_2 = (0; 0, 0, 0, 0, 0, 0, 0, -1) \\ L'_3 = (1; 1, 0, 0, 0, 0, 0, 1, 0) \end{cases} \xrightarrow{T_{127}} \begin{cases} L'_1 = (6; 2, 3, 2, 2, 2, 2, 2, 2) \\ L'_2 = (0; 0, 0, 0, 0, 0, 0, 0, -1) \\ L'_3 = (0; 0, -1, 0, 0, 0, 0, 0, 0) \end{cases}$$

which is (up to a permutation) the first case of (5.23).

We still need to examine the case in which \bar{L}_2 and \bar{L}_3 are adjacent and are given by the second set of equalities in (5.23). Smoothing the union $\bar{L}_2 \cup \bar{L}_3$ to a single irreducible component \bar{Q} we obtain a morphism f_2 with dual graph



Dual graph of f_2

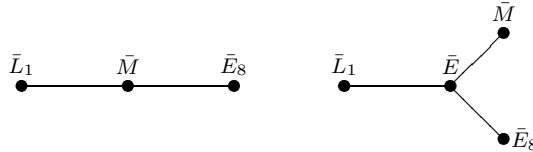
and

$$\begin{cases} (f_2)_* \bar{L}_1 &= (5; 2, 2, 2, 2, 2, 2, 1, 1) \\ (f_2)_* \bar{Q} &= (1; 0, 0, 0, 0, 0, 0, 1, 0) \end{cases}$$

Let $\bar{p} \in \bar{L}_1$ be a point such that $f_2(\bar{p}) \in M$, where $M \subset X$ is the (-1) -curve with divisor class $\ell - e_7 - e_8$. Considering the morphism

$$\mathrm{Sl}_{f_2}(\bar{Q}) \xrightarrow{a} \bar{L}_1$$

we deduce that we may slide \bar{Q} along \bar{L}_1 until the node between these two components reaches the point \bar{p} . When this happens, the image of the limiting position of the image of \bar{Q} contains a point of M . Since the intersection product $(f_2)_*[\bar{Q}] \cdot M$ equals zero, it follows that the image of the limiting position of \bar{Q} must contain M . Thus the limit of the morphism f_2 under this deformation is a morphism f_3 whose dual graph is one of the graphs



Possible dual graphs of f_3

where \bar{E}_8 is mapped to the divisor class e_8 , and the component \bar{E} is contracted by f_3 . The second case happens if the (-1) -curve with divisor class e_8 contains the point $f_2(\bar{p})$. In both cases the point represented by f_3 lies in a unique irreducible component of $\overline{\mathcal{M}}_{0,0}(X, -K_X - K_{X'})$: in the first case

thanks to Proposition 1.17; in the second case thanks to Lemma 2.11 and the fact that the intersection number $(f_2)_*[\bar{L}_1] \cdot E_8$ is one.

We may therefore deform $\bar{L}_1 \cup \bar{E} \cup \bar{M}$ to a unique irreducible component \bar{K} mapped to the divisor K with class $(6; 2, 2, 2, 2, 2, 2, 2) = -2K_X$.

Thus in all cases we found a morphism in the same irreducible component of $\overline{\mathcal{M}}_{bir}(X, -K_X - K_{X'})$ as f whose image contains a nodal integral divisor in $|-2K_X|$. Let $E \subset \overline{\mathcal{M}}_{0,0}(X, -K_X - K_{X'})$ be the subspace consisting of those morphisms containing a component mapped birationally to an irreducible divisor in $-2K_X$. We are going to prove that the space E is connected and contained in the smooth locus of $\overline{\mathcal{M}}_{0,0}(X, -K_X - K_{X'})$. This concludes the proof of the irreducibility of $\overline{\mathcal{M}}_{bir}(X, -K_X - K_{X'})$.

Any morphism $[f : \bar{K} \cup \bar{E}_8 \rightarrow X] \in E$ is determined by its image together with one of the points $f(\bar{K} \cap \bar{E}_8) \in K \cap E_8$. Since $-2K_X \cdot E_8 = 2$, it follows that E has at most two irreducible components.

Suppose E has two irreducible components. Consider the morphism $\varphi : X \rightarrow \mathbb{P}^3$ induced by the linear system $|-2K_X|$. We have already seen that the image is a quadric cone Q and that the morphism is ramified along a smooth curve R which is the complete intersection of Q with a cubic surface. The (-1) -curve E_8 has as image the intersection of Q with a plane which is everywhere tangent to the curve R (and does not contain the vertex of the cone). Let e be one of the intersection points of $\varphi(E_8)$ with R . Projection away from the tangent line L to R at e determines a morphism $\pi_L : R \rightarrow \mathbb{P}^1$ of degree four. Since the genus of R is four, it follows from the Hurwitz formula that the degree of the ramification divisor of π_L is 14. It is immediate to check that π_L ramifies above the tangent plane to Q at e , and that the ramification index is two. It is also immediate that above the plane containing $\varphi(E_8)$ the ramification index is two. We deduce that there are planes in the pencil containing L which are tangent to R and are not the tangent plane to Q at e nor the plane containing $\varphi(E_8)$. Such planes correspond to rational divisors H in $|-2K_X|$ with the property that $H \cap E_8$ consists of the unique point $\varphi^{-1}(e)$. Let $\nu : \bar{H} \rightarrow X$ be a birational morphism whose image is one of the divisors H constructed above. The morphism ν represents a morphism in $\overline{\mathcal{M}}_{bir}(X, -2K_X)$, and since this space is irreducible by assumption, we may deform f to a morphism $\bar{f} : \bar{H} \cup \bar{E}_8 \rightarrow X$, such that $\bar{f}|_{\bar{H}} \simeq \nu$. Thus $\bar{f} \in E$ and it clearly lies in the intersection of the two irreducible components of E . The space E is therefore connected.

Applying Proposition 1.17 we immediately see that all the points of E are smooth in $\overline{\mathcal{M}}_{0,0}(X, -K_X - K_{X'})$, and thus we conclude that the space $\overline{\mathcal{M}}_{bir}(X, -K_X - K_{X'})$ is irreducible. \square

5.2 The Divisor $-K_{X_8} + Q$

We prove now a similar result for the Divisor $-K_{X_8} + Q$.

Lemma 5.2 *Let X be a del Pezzo surface of degree one and suppose that all the rational divisors in $|-K_X|$ are nodal. Let Q be the divisor class of a conic, then the space $\overline{\mathcal{M}}_{bir}(X, -K_X + Q)$ is irreducible.*

Proof. Let $f : \mathbb{P}^1 \rightarrow X$ be a free morphism birational to its image, such that $f_*[\mathbb{P}^1] = -K_X + Q$. As before, we may assume that the image of f contains a general point p . Since there is a one parameter family of deformations of f whose image contains the general point p , we may deform f to a morphism $f' : \bar{C}_1 \cup \bar{C}_2 \rightarrow X$ such that $-K_X \cdot f'_*[\bar{C}_i] = i$. Since p is general and contained in the image of f' and there are no rational cuspidal divisors in $|-K_X|$, it follows that the point represented by f' in $\overline{\mathcal{M}}_{bir}(X, -K_X + Q)$ is smooth.

Our next step is to show that we may assume that $f'_*[\bar{C}_1] = -K_X$. Suppose that $f'_*[\bar{C}_1]$ is a (-1) -curve $L \subset X$. We may choose a standard basis $\{\ell, e_1, \dots, e_8\}$ such that $Q = \ell - e_1$ and thus $-K_X + Q = (4; 2, 1, 1, 1, 1, 1, 1, 1)$. By examining [Ma] Table IV.8, we see that the only possible ways of writing $-K_X + Q$ as a sum of a (-1) -curve C_1 and a nef divisor class C_2 are (up to permutation of the coordinates $2, \dots, 8$):

$$-K_X + Q = \left\{ \begin{array}{ll} (3; 2, 1, 1, 1, 1, 1, 1, 0) & + (1; 0, 0, 0, 0, 0, 0, 0, 1) \\ (2; 1, 1, 1, 1, 1, 0, 0, 0) & + (2; 1, 0, 0, 0, 0, 1, 1, 1) \\ (1; 0, 0, 0, 0, 0, 0, 1, 1) & + (3; 2, 1, 1, 1, 1, 1, 0, 0) \\ (0; -1, 0, 0, 0, 0, 0, 0, 0) & + (4; 3, 1, 1, 1, 1, 1, 1, 1) \\ (1; 1, 1, 0, 0, 0, 0, 0, 0) & + (3; 1, 0, 1, 1, 1, 1, 1, 1) \\ (0; 0, 0, -1, 0, 0, 0, 0, 0) & + (4; 2, 1, 2, 1, 1, 1, 1, 1) \end{array} \right. \quad (5.24)$$

The automorphisms of $\text{Pic}(X)$ of the form T_{1jk} preserve the conic Q , for all $1 < j < k \leq 8$. We use these automorphisms to reduce the number of

cases. We have

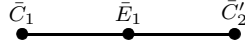
$$\begin{aligned}
T_{167} \left((3; 2, 1, 1, 1, 1, 1, 1, 0) + (1; 0, 0, 0, 0, 0, 0, 0, 1) \right) &= \\
&= (2; 1, 1, 1, 1, 1, 0, 0, 0) + (2; 1, 0, 0, 0, 0, 1, 1, 1) \\
T_{178} \left((1; 0, 0, 0, 0, 0, 0, 1, 1) + (3; 2, 1, 1, 1, 1, 1, 0, 0) \right) &= \\
&= (0; -1, 0, 0, 0, 0, 0, 0, 0) + (4; 3, 1, 1, 1, 1, 1, 1, 1) \\
T_{123} \left((1; 1, 1, 0, 0, 0, 0, 0, 0) + (3; 1, 0, 1, 1, 1, 1, 1, 1) \right) &= \\
&= (0; 0, 0, -1, 0, 0, 0, 0, 0) + (4; 2, 1, 2, 1, 1, 1, 1, 1)
\end{aligned}$$

We therefore only need to consider the first, third and fifth case in list (5.24). We reduce the first and third case to the fifth one.

If \bar{C}_1 is mapped to the divisor class $(3; 2, 1, 1, 1, 1, 1, 0)$, then we consider the morphism

$$\mathrm{Sl}_{f'}(\bar{C}_2) \xrightarrow{a} \bar{C}_1$$

Since $f'|_{\bar{C}_2}$ is a free morphism, a is dominant. Let $\bar{p} \in \bar{C}_1$ be a point such that $p := f'(\bar{p})$ lies on the (-1) -curve with divisor class e_1 . Let $g : \bar{C}_1 \cup \bar{E}_1 \cup \bar{C}'_2 \rightarrow X$ be a morphism such that $a(g) = \bar{p}$. By construction, the dual graph of g is



Dual graph of g

where \bar{E}_1 is mapped to the divisor class e_1 and \bar{C}'_2 to the divisor class $\ell - e_1 - e_8$. We now smooth the union $\bar{C}_1 \cup \bar{E}_1$ to a single irreducible component. Thus, after a permutation of the indices, we reduced to the fifth case in (5.24).

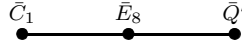
If \bar{C}_1 is mapped to the divisor class e_1 , then we proceed similarly: break \bar{C}_2 into a component mapping to the divisor class $(3; 2, 1, 1, 1, 1, 1, 0)$ adjacent to \bar{C}_1 , and a component mapped to the divisor class $\ell - e_1 - e_8$. Smoothing the union of the component \bar{C}_1 with the component mapped to $(3; 2, 1, 1, 1, 1, 1, 0)$ reduces us to the fifth case in (5.24).

Suppose therefore that the component \bar{C}_1 is mapped to $\ell - e_1 - e_2$ and the component \bar{C}_2 is mapped to $3\ell - e_1 - e_3 - \dots - e_8$. As above, we may deform the morphism f' to a morphism g so that the component \bar{C}_2 breaks into a component \bar{E}_2 adjacent to \bar{C}_1 and mapped to e_2 , and into a component \bar{C}'_2 mapped to the divisor class $-K_X$. Smoothing the union $\bar{E}_2 \cup \bar{C}'_2$ to a

single irreducible component, we obtain a morphism $g' : \bar{C}_1 \cup \bar{Q} \rightarrow X$, where \bar{C}_1 is mapped to $-K_X$ and \bar{Q} is mapped to Q . Note that we have at the moment no control over which rational divisor in the linear system $|-K_X|$ the component \bar{C}_1 is mapped to. Remember that with our choice of standard basis we have $Q = \ell - e_1$. We may write $Q = (\ell - e_1 - e_8) + e_8$, and since $-K_X \cdot e_8 = 1$, there is a unique point \bar{c} of \bar{C}_1 whose image $c \in X$ lies in E_8 , the (-1) -curve on X with divisor class e_8 . Considering the dominant morphism

$$\mathrm{Sl}_{g'}(\bar{Q}) \xrightarrow{a} \bar{C}_1$$

we may find a morphism $h : \bar{C}_1 \cup \bar{E}_8 \cup \bar{Q}' \rightarrow X$ such that $a(h) = \bar{c}$. The dual graph of h is



Dual graph of h

Smoothing the components $\bar{C}_1 \cup \bar{E}_8$ to a single irreducible component \bar{K}' we obtain a morphism $h' : \bar{K}' \cup \bar{Q}' \rightarrow X$ such that $Q' := h'_*[Q'] = \ell - e_1 - e_8$ and $h'_*[\bar{K}'] = -K_X + e_8 = -K_{X'}$, where X' is the del Pezzo surface obtained from X by contracting the (-1) -curve E_8 .

Let $H \subset \overline{\mathcal{M}}_{bir}(X, -K_X + Q)$ be the space of morphisms whose image contains E_8 and an integral rational divisor in $|-K_{X'}|$. We have a dominant morphism

$$\pi : H \rightarrow \overline{\mathcal{M}}_{bir}(X, -K_{X'})$$

whose fibers have length two, since $-K_{X'} \cdot Q' = 2$. It follows that H has at most two irreducible components. Note that the fibers of π over the general point of $\overline{\mathcal{M}}_{bir}(X, -K_{X'})$ are smooth points of $\overline{\mathcal{M}}_{bir}(X, -K_X + Q)$. It follows that the space $\overline{\mathcal{M}}_{bir}(X, -K_X + Q)$ itself has at most two irreducible components, and is irreducible if H is. We prove that if H is reducible, then we can find a smooth point of $\overline{\mathcal{M}}_{bir}(X, -K_X + Q)$ lying in the intersection of the two components of H . This is enough to imply that $\overline{\mathcal{M}}_{bir}(X, -K_X + Q)$ is irreducible.

Suppose thus that H has two irreducible components. Let $\varphi : X \rightarrow \mathbb{P}^2$ be the morphism induced by the linear system $-K_{X'}$. The morphism φ is the contraction of E_8 to X' followed by the anticanonical double cover of \mathbb{P}^2 ramified above a smooth plane quartic R . The image of Q' in \mathbb{P}^2 is a conic \tilde{Q} containing the image of E_8 and everywhere tangent to R . The image of K' is a tangent line to R . To conclude it is enough to find a line in \mathbb{P}^2 which is tangent to both R and \tilde{Q} at a point not on R . The dual curve of R is a plane curve of degree twelve and the dual curve of \tilde{Q} is a plane conic. Thus

they meet along a scheme of length 24 and they are tangent at the points corresponding to the points where R and \tilde{Q} are tangent. Since there are four such points, it follows that we may find a line which is tangent to R and \tilde{Q} at distinct points. Such a line corresponds to a point in the intersection of the two components of H . Using Proposition 1.17 it is easy to check that this point is smooth in $\overline{\mathcal{M}}_{bir}(X, -K_X + Q)$. This completes the proof of the lemma. \square

6 Conclusion

6.1 The Irreducibility of $\overline{\mathcal{M}}_{bir}(X_\delta, \beta)$

We are now ready to prove the main theorems of this article.

Theorem 6.1 *Let X_δ be a del Pezzo surface of degree $9 - \delta \geq 2$. The spaces $\overline{\mathcal{M}}_{bir}(X_\delta, \beta)$ are irreducible or empty for every divisor $\beta \in \text{Pic}(X_\delta)$.*

Proof. Suppose $\overline{\mathcal{M}}_{bir}(X_\delta, \beta)$ is not empty. Then β is represented by an effective integral curve on X_δ .

If β is not nef, then it follows that $\beta^2 < 0$. We deduce that β is a positive multiple d of a (-1) -curve. If $d = 1$, then $\overline{\mathcal{M}}_{bir}(X_\delta, \beta)$ consists of a single point. If $d > 1$, then the space $\overline{\mathcal{M}}_{bir}(X_\delta, \beta)$ is empty. In this case, the space $\overline{\mathcal{M}}_{0,0}(X_\delta, \beta)$ is irreducible, since it is dominated by the space of triples of homogeneous polynomials of degree d in two variables.

Suppose now that β is a nef divisor. Thanks to Theorem 4.5, we simply need to check that on a del Pezzo surface of degree at least two, the spaces $\overline{\mathcal{M}}_{bir}(X_\delta, \beta)$ are irreducible for all effective integral divisor classes β such that $-K_{X_\delta} \cdot \beta$ equals two or three. The divisors of degree two on X_δ are the conics and, if $\delta = 7$, the divisor $-K_{X_7}$. If β is a conic, then $\overline{\mathcal{M}}_{bir}(X_\delta, \beta)$ is isomorphic to \mathbb{P}^1 . If $\beta = -K_{X_7}$, then $\overline{\mathcal{M}}_{bir}(X_7, -K_{X_7})$ is isomorphic to a smooth plane quartic curve, Proposition 2.8.

The nef divisors of degree three on X_δ are $-K_{X_6}$ and ℓ , where X_6 is a del Pezzo surface of degree three dominated by X_δ and ℓ is part of a standard basis $\{\ell, e_1, \dots, e_\delta\}$. The first case is treated in Proposition 2.7, the second case is treated in Theorem 3.3. This concludes the proof of the theorem. \square

Theorem 6.2 *Let X_8 be a general del Pezzo surface of degree one. The spaces $\overline{\mathcal{M}}_{bir}(X_8, \beta)$ are irreducible or empty for every divisor $\beta \in \text{Pic}(X_8)$, with the unique exception of $\beta = -K_{X_8}$. The space $\overline{\mathcal{M}}_{bir}(X_8, -K_{X_8})$ is a reduced scheme of length twelve.*

Proof. Proceeding as before, we only need to prove the irreducibility of $\overline{\mathcal{M}}_{bir}(X_8, \beta)$ for the nef divisors of anticanonical degree two or three. The nef divisor classes on X_8 which are not ample, are the pull-back of nef divisor classes from del Pezzo surfaces of larger degree. Thus we only need to consider ample divisor classes of anticanonical degree two or three.

The only ample divisor of degree two is $-2K_{X_8}$ and the mapping space $\overline{\mathcal{M}}_{bir}(X_8, -2K_{X_8})$ is irreducible thanks to Theorem 2.10.

The ample divisor classes of degree three on X_8 are $-3K_{X_8}$, $-K_{X_8} - K_{X_7}$ and $-K_{X_8} + Q$, where X_7 is a del Pezzo surface of degree two dominated by X_8 and Q is the divisor class of a conic. The space $\overline{\mathcal{M}}_{bir}(X_8, -3K_{X_8})$ is irreducible thanks to Theorem 2.12. The space $\overline{\mathcal{M}}_{bir}(X_8, -K_{X_8} - K_{X_7})$ is irreducible thanks to Lemma 5.1. The space $\overline{\mathcal{M}}_{bir}(X_8, -K_{X_8} + Q)$ is irreducible thanks to Lemma 5.2. This concludes the proof of the theorem. \square

Remark. The genericity assumption on X_8 in the statement of the Theorem 6.1 can be made more explicit. Our argument requires the surface X_8 to have only nodal rational divisors in $|-K_{X_8}|$ and the space $\overline{\mathcal{M}}_{bir}(X_8, -2K_{X_8})$ to be irreducible. This last condition in turn is certainly satisfied (cf. Theorem 2.10 and its proof) if the ramification curve $R \subset \mathbb{P}^3$ of the morphism $\varphi : X_8 \rightarrow \mathbb{P}^3$ induced by $-2K_{X_8}$ does not admit planes $P \subset \mathbb{P}^3$ transverse to the image of φ and intersecting R along a divisor of the form $3((p) + (q))$.

As a corollary of the above Theorems, we deduce the irreducibility of the Severi varieties of rational curves on the del Pezzo surfaces. Let β be a divisor class in $\text{Pic}(X_\delta)$ and let $V_{0,\beta} \subset |\beta|$ be the closure of the set of points corresponding to integral rational divisors. We call $V_{0,\beta}$ the Severi variety of rational curves on X with divisor class β .

Corollary 6.3 *Let X_δ be a del Pezzo surface of degree $9 - \delta \geq 2$. The Severi varieties $V_{0,\beta}$ of rational curves on X_δ are either empty or irreducible for every divisor $\beta \in \text{Pic}(X_\delta)$.* \square

Corollary 6.4 *Let X_8 be a general del Pezzo surface of degree one. The Severi varieties $V_{0,\beta}$ of rational curves on X_8 are either empty or irreducible for every divisor $\beta \in \text{Pic}(X_8)$, with the unique exception of $\beta = -K_{X_8}$.* \square

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